Critical review of the 1. Stokes' problem and consequences for mixed turbulent/laminar flow

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Abstract

The "1. Stokes' problem", the "suddenly accelerated flat wall", is the oldest application of the Navier-Stokes equations. Stokes' solution of the "problem" does not comply with the mathematical theorem of Cauchy and Kowalewskaya on the "Uniqueness and Existence" of solutions of partial differential equations and violates the physical theorem of minimum entropy production/dissipation of the Thermodynamics of Irreversible Processes. The result includes very high local shear stresses and dissipation rates. That is of special interest for the theory of turbulent and mixed turbulent/laminar flow. A textbook solution of the "1. Stokes Problem" is the Couette flow, which has a constant sheer stress along a linear profile. A consequence is that the Navier-Stokes equations do not describe any S-shaped part of a turbulent profile found in any turbulent Couette experiment. The paper surveys arguments referring to that statement, concerning the history of >150 years. Contrary to this there is always a Navier-Stokes solution near the wall, observed by a linear part of the Couette profile. There a turbulent description (e.g. by the logarithmic law-of-the-wall) fails completely. That is explained by the minimum dissipation requirement together with the Couette feature τ = const. The local co-existence of a turbulent zone and a laminar zone near the wall is stable and observed also at high Reynolds-Numbers.

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1. Introduction

The "1. Stokes' problem" is one of the oldest applications of the Navier-Stokes equations. Stokes' solution does not comply with the theorems of Cauchy and Kowalewskaya on the "Existence and Uniqueness" of solutions of partial differential equations and with the theorem of minimum entropy production/dissipation. A consequence is that the Navier-Stokes equations do not describe turbulence observed in the Couette flow experiment.

Stokes was the first to focus on "problems" of the most important hydrodynamic equations published in his century in 1851 by him. A "corpuscular" fluid character (e.g. kinetic theory of gases) was identified 9 years later, turbulence was described 33 years later, and the thermodynamics of irreversible processes are 85 years later than the Navier Stokes Equations and Stokes' "problems".

The "1. Stokes solution" is discussed with two different "problems" – the "suddenly accelerated flat plate" and the "suddenly accelerated Couette flow". Both solutions start with infinitely high gradients at the start of the "jerk", therefore are not twofold differentiable and lead to extremely high dissipation rates. That is mathematically and physically not to be accepted.

The problems of the "problem" are the initial values of the solution. Both solutions have the same problem near the wall but differ significantly off the wall and at greater time intervals. The "suddenly accelerated flat wall" has no unique Navier-Stokes solution. The "suddenly accelerated Couette flow" has a stationary Navier-Stokes solution valid for arbitrary long time intervals. After long time intervals that solution is stationary, linear, laminar and valid for arbitrary Reynolds numbers - and can be discussed being unique. That includes the Navier-Stokes equations not describing non-linear, non-laminar, turbulent S-shaped Couette profiles observed in the experiment.

Generally, the Navier-Stokes equations are discussed as the most important hydrodynamic equations of the last two centuries. The statements above include critical restrictions for the application of the Navier-Stokes equations for turbulent flow. That requires consequences.

The first consequence is a review of arguments over > 170 years. Stokes himself (1851) mentions limitations of the theory to low flow speeds and Rayleigh (1911) explicitly to "infinitely low" flow speeds. Reynolds proposed "averaged" equations for turbulence, named after him. 1925 the "universal" logarithmic "law-of-the-wall" (Prandtl, v. Kárman, Taylor) for "averaged" turbulent solutions was proposed without any use of the Navier-Stokes equations. V. Kárman (1924) and Oertel jun. (2015) criticized the "instability theory" of the Navier-Stokes equations for turbulence. Olga Ladyzhenskaya published 1963 the excellent “Mathematical Theory Of Viscous Flow”, which became the basis for defining the Navier-Stokes equations a “1 Mio. US $ Millenium problem (Fefferman 2000). Some citations are very critical (Lord Raleigh 1911, Fefferman 2000)”

The next consequences to be discussed arrive directly from the statements above. If the solution of a part of a fluid dynamics problem does not comply with the theorem of “Uniqueness and Existence” that part of the solution is wrong. If the solution of a part of a fluid dynamics problem does not comply with the “Minimum dissipation” requirement that part of the solution is not observed in the experiment.

That may be surprising, we recommend the first statement of Olga Ladyzhenskaya in Chapter 2.2.

The 1. Stokes problem has a final stationary linear and laminar solution for the Couette flow, valid for arbitrary time intervals. That solution is “unique” – but not observed in the experiment. With high Reynolds numbers it does not comply with the minimum dissipation requirement.

The “universal” logarithmic “law-of-the-wall” describes the central turbulent flow zone of the Couette experiment. It is a hypothesis (a very successful one) and not at all based on any partial differential equation. At a quantified near wall distance, it fails completely mathematically (with negative flow values), and physically due to not fulfilling the minimum dissipation requirement. Here the linear Navier-Stokes solution is observed again and always.

2. The 1. Stokes' problem

2.1 Stokes' solution

The "suddenly accelerated" flat plate, the "1. Stokes' Problem", was solved by Stokes in 1851, the following description is according /1/. The partial differential equation near the plane wall is

$$\frac{\partial u}{\partial t} = \nu \cdot \frac{\partial^2 u}{\partial y^2} \tag{1}$$

identical to the heat transfer equation. By introducing the dimensionless variable η

$$\eta = \frac{y}{2\sqrt{\nu \cdot t}} \tag{2}$$

and by setting with U_o = speed of the "jerk"

$$u = U_o \cdot f(\eta) \tag{3}$$

results the ordinary differential equation for $f(\eta)$

$$f" + 2\eta \cdot f' = 0 \tag{4}$$

The initial boundary conditions are $f = 1$ at the wall (at $\eta = 0$) and $f = 0$ at $\eta = \infty$

The result is $$u = U_o \left[1 - \frac{2}{\sqrt{\pi}} \int_o^\eta e^{-\eta^2} \cdot d\eta\right] \tag{5}$$

shown in Fig. 1

The wall shear stress τ_o is /2/, /19/ $$\tau_o = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{\rho U_o}{\sqrt{\pi}} \sqrt{\frac{\nu}{t}} \tag{6}$$

With t → o gradient, shear stress and dissipation near the wall approach infinite values. That is mathematically and physically not to be accepted. We will discuss both aspects.

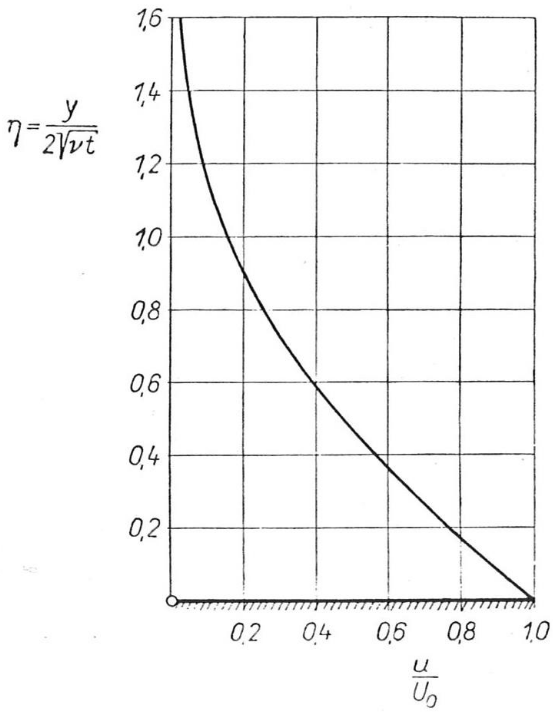


Fig. 1: Solution of the 1. Stokes' problem, Schlichting /1/

The textbook solution of Schlichting /1/ in Fig. 1 shows for $\eta = 0$ varying values of u in the range

$$0 < u(o) < U_o$$

The textbook of K. Wieghardt /2/ shows the drawing Fig. 2 of the same solution at $\eta = 0$ with a fixed value of

$$u(o) = U_o$$

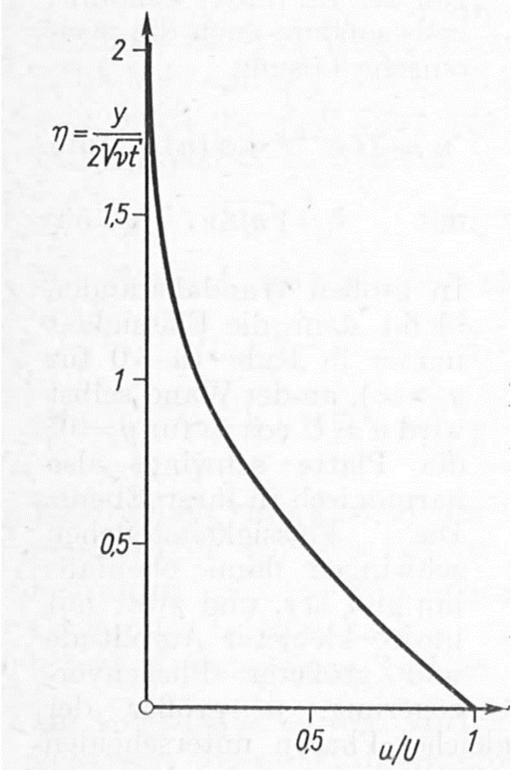


Fig. 2: Solution of the 1. Stokes' problem, Wieghardt /2/

Schlichtings Fig. 1 with varying u-values at $\eta = 0$ (see different thickness of the horizontal axis in Fig. 1) gives the impression that it describes the situation "during" the jerk. Wieghardts Fig. 2 with a constant u-value u = U_o at $\eta = 0$ describes the solution "after completing" the jerk. Wieghardts description is the correct one, the initial condition f = 1 at $\eta = 0$ corresponds to u = U_o, but it does not describe the jerk itself.

The difference of Fig. 1 and Fig. 2 at $\eta = 0$ in two excellent textbooks points at the question of "Uniqueness and Existence".

The solution of the "accelerated flat wall" has no final mathematical and physical result. Even "smoothing" the initial "jerk", will not simplify the question of "Existence and Uniqueness".

By introducing a "fixed" wall opposite the "accelerated" wall at a distance h the calculations describe the "sudden" start of the Couette flow.

The boundary conditions of the differential equation are now $f = 1$ at the wall (at $\eta = 0$) and $f = 0$ at $\eta = \eta_1 = \frac{h}{2}\sqrt{\nu \cdot t}$ defining the dimensionless distance of the two walls.

The application of the solution of the 1. Stokes' problem to a "sudden accelerated" Couette flow experiment is more complicated than Eq. (5) and is described by Schlichting /1/. Fig. 3 shows the resulting flow profiles after the sudden "jerk". The first profiles are "similar", the final profile is the well-known stationary, linear, laminar Couette flow profile.

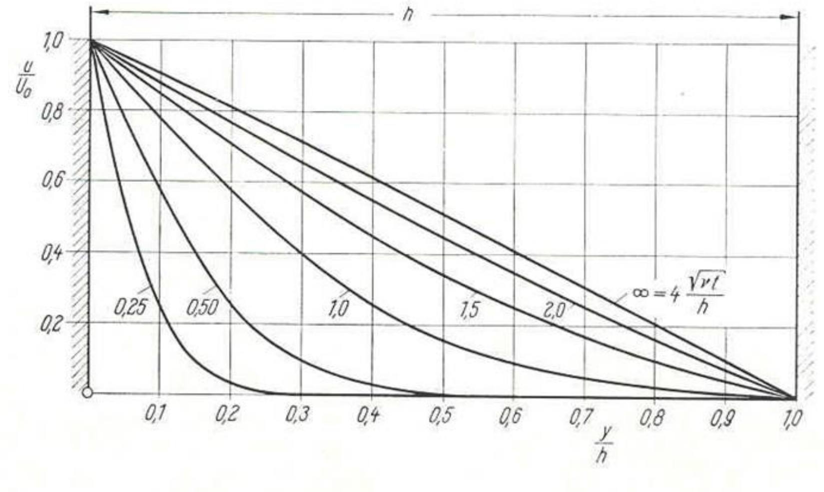


Fig. 3: "Jerky" acceleration of the Couette flow /1/

2.2 Existence and uniqueness theorem

O. Ladyzhenskaya /3/ warns the reader of the difference of "what is understood by the solution of a problem and what it means to solve a problem". That differs "from which he is familiar". She requires a "precise analysis of these matters", especially a "rigorous" mathematical analysis of the solution of boundary-value problems for the Navier-Stokes equations /3/.

Olga Ladyzhenskaya is right. Stokes' "solution" has a textbook history > 150 years, we have to focus the "problem".

Solutions of nonstationary boundary value problems – like the 1. Stokes problem "require"

- "small" Reynolds numbers at the initial instant of time /3/
- external forces being derived from a potential /3/
- "a certain smoothness of the initial field and of the external forces" /3/

That includes that the relevant functions of the external forces and of the solution are

- square summable and twofold differentiable. /3/ /4/

It is evident that the result of the 1. Stokes solution does not comply with those requirements. If the right side of Eq. (1) is not differentiable the left side of Eq. (1) cannot be integrated.

The 1. Stokes' solutions for the "suddenly accelerated wall" and for the "suddenly accelerated Couette flow" give non-mathematical infinitely high gradients near the wall and non-physical infinite dissipation rates.

The substitution of a variable η (Eq. (2)) in a partial differential equation Eq. (1) is a mathematical standard procedure but might create further problems. The substitution of the variable u by the variable η simplifies the solution but fixes the mathematical form of an unknown variable u.

The substitution of a variable in Eq. (2) is pioneered by Stokes, himself citing Cauchy in his famous ("celebrated" /19/) publication /18/. The focus of Stokes' 90-pages publication are slow moving pendulums in a viscous fluid.

The unknown function u is defined u = u (y, t) with the independent (geometric) variable y and the independent (time) variable t. The substituted variable η itself is defined $\eta = \eta\left(\frac{y}{\sqrt{t}}\right)$ and that is not defined near the "accelerated" wall at y -> 0 and t -> 0.

There are remarkable differences with the solution of the "suddenly accelerated Couette flow". There is a final, stationary, linear result valid for arbitrary long-time intervals. That can be considered as being unique for the stationary result.

We do not need a final discussion of that aspect. If the final (stationary) result is unique, there is no turbulent result at all. If not, there is no existence of a unique turbulent solution, either.

2.3 Minimum condition for entropy-production/dissipation

The theorem of minimum entropy production/dissipation of the thermodynamics of irreversible processes states that non-equilibration thermodynamic processes proceed in a manner in which the entropy production/dissipation becomes minimal /5/.

According to Helmholtz and Rayleigh the dissipation has a minimum behaviour, described by Wieghardt /2/. This is based on the Navier-Stokes Equations and therefore valid only for Navier-Stokes solutions. The generalization to different fluid flow mechanisms resulted nearly hundred years later by the thermodynamics of irreversible processes /5/.

The theory of irreversible thermodynamic processes is associated with the names of Onsager (Nobel Prize 1966), Casimir, Eckart, Meixner, de Groot, and Prigogine (Nobel Prize 1977). The minimum requirement is mentioned as “Prigogine-Prinzip” in the German Brockhaus Enzyklopädie /10/ and as “Rayleigh-Onsager principle of least dissipation or principle of minimum entropy production” in the British Encyclopaedia Britannica /7/.

In fluid dynamics theory the minimum principle has not always been accepted. Malkus/Busse /8/ postulate a general maximum dissipation for turbulence. Klimantovich /9/ states a general minimum dissipation. Both statements do not comply.

It is surprising, to find a new paper (2021) “proposing a principle of Maximum Entropy Production (MEP)” – without discussing origin and consequences of that “principle” /16/.

We consider the shear stress near the wall τ_0. The wall shear stress in Fig. 1 is

$$\tau_0 = \mu \cdot \frac{du}{dy}(y = 0)$$

The textbook solution give

$$\tau_0 \sim \frac{1}{\sqrt{t}} \qquad \tau_0 \to \infty \qquad with\, t \to 0$$

At the start of the “sudden jerk” there is an infinitely high shear stress, an infinitely high gradient and an infinitely high dissipation near the wall.

This is an impossible physical result. If one models the “jerk” under the assumptions of the kinetic theory of gases (or another corpuscular fluid model), finite values for shear stress and dissipation result.

From a gas molecule layer of thickness *l* (*l* = average free path length), in the first time period $l/\bar{c}$ after the very sudden start of the plate, $\frac{l \cdot n}{6}$ molecules per surface unit (n = number per volume) touch the plate and leave it after an elastic collision with the additional velocity components 2 U_o (U_o = plate velocity) and the additional momentum components 2 U_o m (m = molecular weight). Therefore, per plate surface and time unit, the momentum is changed by τ_0

$$\tau_0 \left(o < t < \frac{l}{\bar{c}}\right) = (2u_o m)\left(\frac{ln}{6}\right) \Big/ \frac{l}{\bar{c}}$$

$$= \frac{1}{3} \rho \, u_o \bar{c}$$

In contrast to the result of the 1. Stokes' solution, the wall shear stress is finite. No dissipation has taken place until the time $t = \frac{l}{\bar{c}}$ after the start of the "jerk".

The mathematical solution of the Navier-Stokes equations results in a high, local dissipation. A fluid model with suitable "corpuscular" properties results in a physically reasonable solution. But such a model is not described by the Navier-Stokes equations. F. Durst formulates this drastically: "Explaining viscosity with internal "fluid friction" is physically incorrect!" /10/.

The logarithmic turbulent flow profile, described in Chap. 3.2 for the Couette flow, leads to a gradient and a dissipation rate becoming continuously lower with greater wall distance. The logarithmic flow profile characterises a non-local turbulent flow mechanism with the input and transport of mechanical energy from the wall to the locus of dissipation inside the flow. The non-local character of the turbulent zone 1 in the centre in Fig. 5 differs completely from the local character of the laminar zone 3 near the wall in Fig. 5.

In a steady state Couette experiment laminar zones near the wall and turbulent flow zones in the centre are stable, characterized by the minimum dissipation condition. In /11/ that is described by a "corpuscular" model for a mixed laminar/turbulent Couette experiment by the Navier-Stokes equations and the hypothesis of the logarithmic "law-of-the-wall". The Navier-Stokes mechanism for the viscous sublayer, is limited to a dimensionless wall-distance $y^+ < 2,5$ /11/. The description by the logarithmic law-of-the-wall, the turbulent zone 1 in Fig. 5 is extended above a dimensionless wall-distance $y^+ > 67,5$ /11/.

These theoretical values can be compared with $y^+ < 5$ and $y^+ > 60$ in the experimental literature /1/. Newer Laser-Doppler experiments are cited by Durst /10/ with $y^+ < 2$).

3. Laminar theory mixed turbulent/laminar reality

3.1 Survey of 170 years history

The survey aspect of the paper covers the history of >150 years application of the Navier-Stokes equations for describing turbulence. The Navier-Stokes equation and Stokes' theory are published 1851, probably the most important and most discussed hydrodynamic equations of the last two centuries.

32 years later turbulence was described, the first experimental description by O. Reynolds in 1884.

Reynolds theory on his experiments focused on simplification by time-averaged solutions, the so-called "Reynolds Averaged Navier-Stokes Equations", well known as R.A.N.S. Consequence: no time dependent solutions and the well-known "closure problem" of the equations – unsolved until today.

A very important step in understanding turbulence was 1924 by Prandtls mixing lengths hypothesis (together with Taylor and v. Kármán) resulting in the "universal logarithmic law of the wall", a well-known and well researched empirical hypothesis, without any use of the Navier-Stokes equations.

1924 came the "instability analysis" of the Navier-Stokes equation (Orr, Sommerfeld, criticized by v. Kármán). It is disputed that an "instability" of a laminar flow solution does not mean describing the turbulent flow solution correctly (Schlichting 1982) /1/. V. Kármán criticises Sommerfeld (1924). Oertel jr. (2015) mentions the "discussion that turbulence has nothing to do with instability" /17/.

Other flow mechanisms are discussed with statistical and other mathematical models like "strange attractors", bifurcation, chaotic behaviour, "continuous transition to chaos via an infinite cascade of bifurcations". The models display chaotic dynamics, "but whether their chaos is an accurate reflection of the kinds of erratic behaviour found in experimental and observational data is an unresolved question in most instances" (Guckenheimer 1981).

There is an old list of critical remarks in the history. Important aspects of survey of published statements are the following.

Stokes /18/ and Rayleigh /19/ mentioned limitations of Navier-Stokes solutions to low flow speeds. Rayleigh mentioned explicitly “infinitely low flow speeds” /19/.

Olga Ladyzhenskaya /3/ recommends to use the equations only with low Reynolds numbers and states: “It is hardly possible to explain the transition laminar to turbulent within the classical Navier-Stokes theory” (1963).

Oertel jr. (2015) mentions the “discussion that turbulence has nothing to do with instability” /17/.

The flow resistance during transition increases by more than one order of magnitude, consequently the transition has to be “forced” (Durst 2006) /10/.

3.2 No compliance with turbulence

For Couette flow the 1. Stokes solution based on a Navier-Stokes solution gives a final laminar, linear, stationary velocity profile Fig. 3 /1/. Contradicting that result Fig. 4 shows experimental laminar profiles at Re = 1200 and two different experimental turbulent flow profiles at Re = 2900 and 34000.

The Navier-Stokes solution Fig. 3 is simple, linear, laminar - but not physically reasonable at higher Re-number.

Unique to Couette flow, there is a constant shear stress along the complete flow profile, being laminar, turbulent or both /1/.

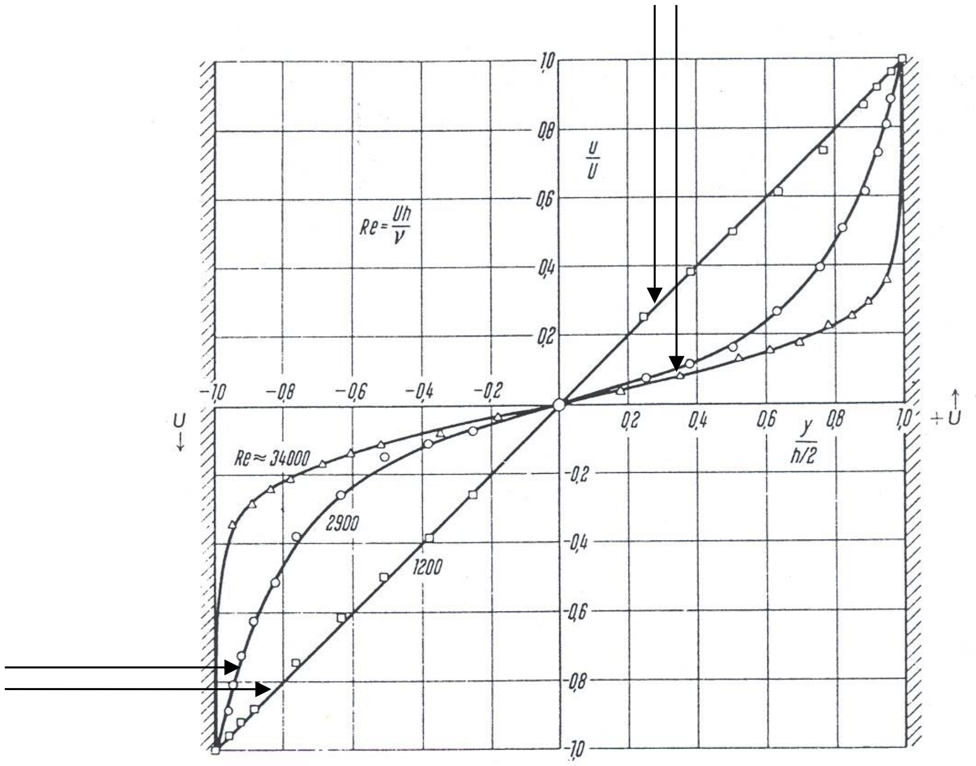


Fig. 4: Velocity profiles of Couette flow /1/

One can simplify the description of the Couette flow profile by three different zones, shown in Fig. 5.

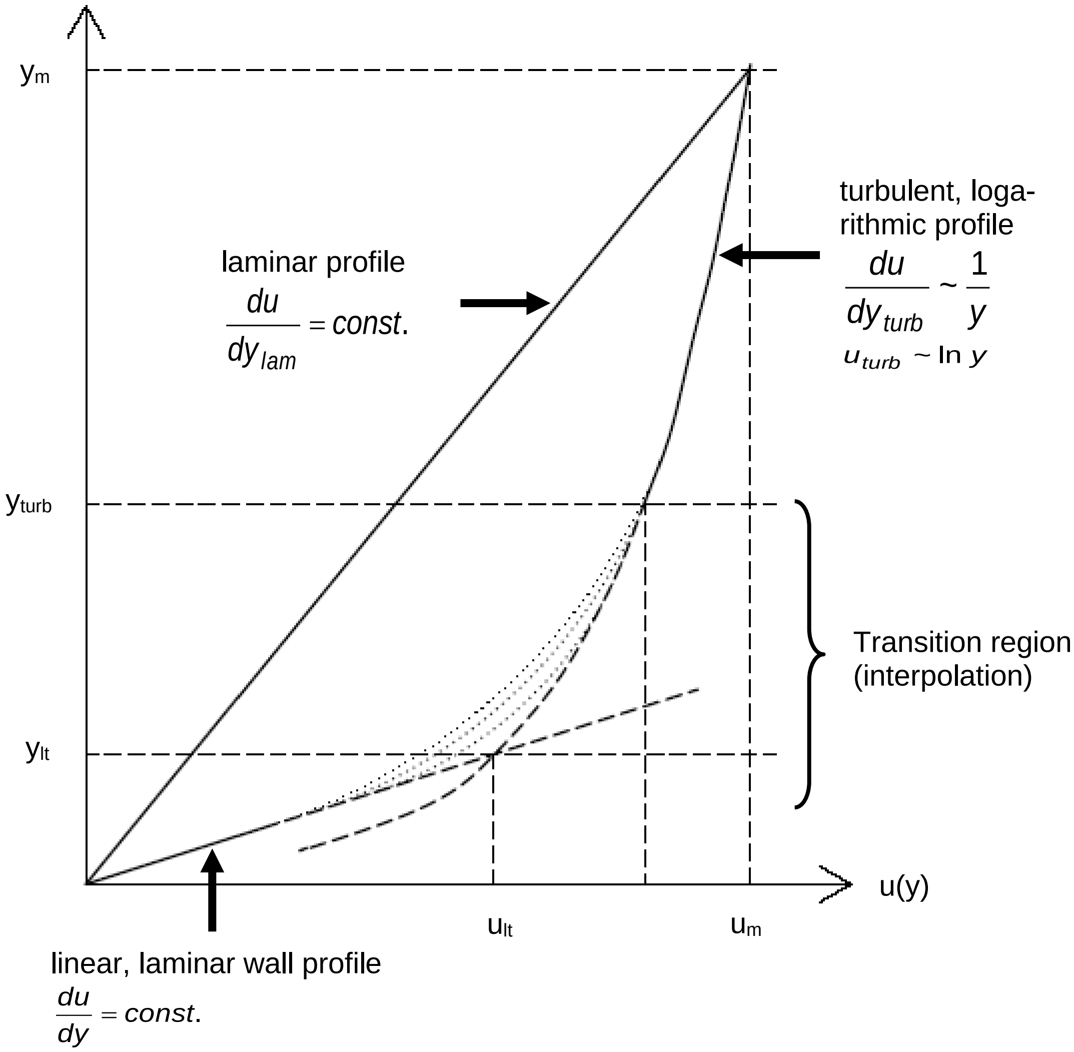


Fig. 5: Model assumption for the Couette flow

Zone 1: The turbulent zone, logarithmic flow profile

Zone 2: The transition zone

Zone 3: The linear (laminar) near-wall-zone

The description of zone 1 can be based on the “universal logarithmic law of the wall” which is based on Prandtl’s and v. Kármán’s hypotheses /1/ with

$$\tau \;=\; \rho\chi^2 y^2 \left(\frac{du}{dy}\right)^2 \tag{7}$$

with $\chi \approx 0{,}4,$ (v. Kármán constant), contradicting a linear Couette profile.

Fig. 4, Fig. 6, Fig. 7 show empirical data, given by Schlichting /1/ and Durst /10/ for low and high Re-numbers and different wall distances.

The resulting logarithmic part of the profiles is fitted with experimental data to

$$\varphi(\eta) = 2.5 \ln \eta + 5{,}5 \quad \text{/1/} \tag{8a}$$

or

$$\varphi(\eta) = 2{,}47 \ln \eta + 5{,}17 \text{ /10/} \tag{8b}$$

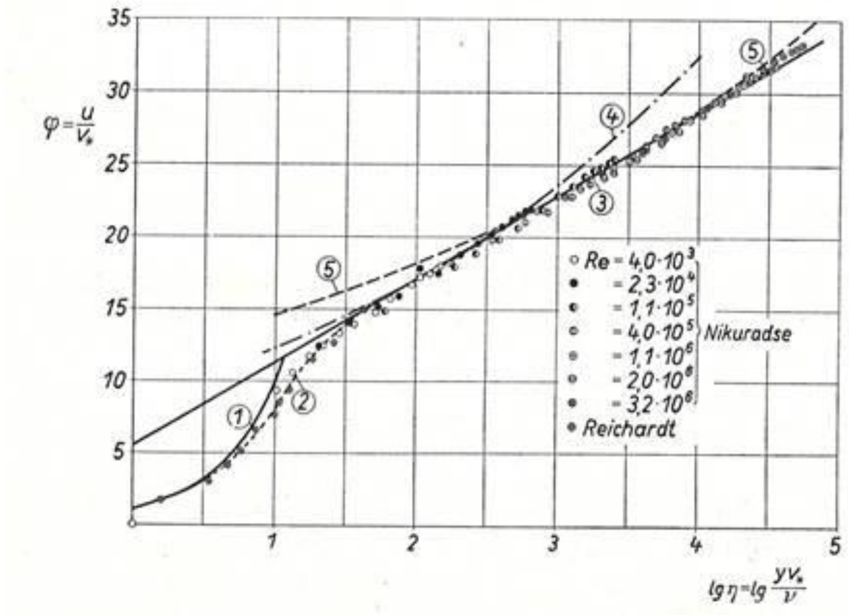
with

$$\varphi = u^+ = \frac{u}{\sqrt{\frac{\tau_0}{\rho}}} \tag{9}$$

as so-called “dimensionless shear stress speed” and

$$\eta = y^+ = \frac{y \cdot \sqrt{\frac{\tau_0}{\rho}}}{\frac{\mu}{\rho}} \tag{10}$$

as so-called “dimensionless wall distance” /1/.



(1) $\varphi = \eta$, laminar
(2) Übergang laminar-turbulent
Re < 10^5

Fig. 6: Universal logarithmic velocity profile (law-of-the-wall) Re < 10^6, Schlichting /1/

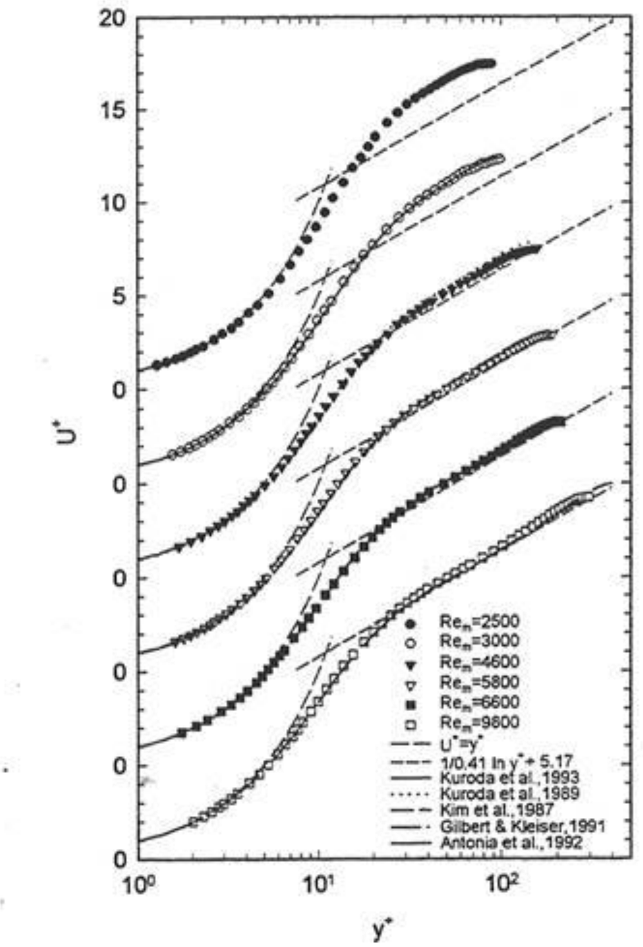


Fig. 7: Universal logarithmic velocity profile (law-of-the-wall) Laser-Doppler measurements, Re < 10^4, Durst /10/

At low dimensionless wall distances, the logarithmic “law-of-the-wall” does not comply with experimental data. Eq. 8a, 8b is based on a constant shear stress τ according Eq. 7. The integration to the final logarithmic formula includes integration constants which are fitted to the experiment /1/. At zero wall distance $y^+ = 0$ results $\varphi = -\infty$. Consequently, Eq. 8a, b are completely wrong near to the wall.

The limitations of the logarithmic law are described by Drescher /11/ and discussed by Platzer /12/, Wieghardt /2/ and Pope /13/.

We shall keep in mind, that the logarithmic “law-of-the-wall” is a hypothesis (a very successful one). The experimental data summarize, that it cannot be used below $y^+ < 60$ (Fig. 6, Fig. 7).

Zone 3:
The viscous sublayer (and the linear profile of the laminar Couette flow) are described perfectly the Navier-Stokes equations, the Newton-Stokes friction resulting in

$$\tau = \mu \cdot \frac{du}{dy} \tag{11}$$

exactly

$$\tau_{ij} = \mu \left(\frac{\vartheta u_j}{\vartheta x_i} + \frac{\vartheta u_i}{\vartheta x_j}\right) \tag{11a}$$

The linear near-wall profile is presented in Fig. 6, Fig. 7 in the simple (but curved) dimensionless form $u^+ = y^+$. Fig. 6, Fig. 7 show that the Stokes friction only describes experimental results below $y^+ \approx 5$.

Zone 2:
For the transition zone we have little information. Reliable experimental data exist but no theoretical model. The range of that zone is $5 \leq y^+ \leq 60$.

The logarithmic “law-of-the-wall”, used to describe turbulent flow, is based on the “corpuscular” mixing length hypothesis (Prandtl, v. Kármán, Taylor) and leads to a “non-local” description. The logarithmic “law-of-the-wall” does not comply near the wall due to a mathematical singularity. The restrictions are described in reference /11/, /12/, /2/, /13/.

A “corpuscular” non-local model, based on the requirement of minimum dissipation, is used by Drescher /11/ to explain the co-existence of laminar and turbulent zones and their transition.

Maximum turbulent dimensions and a calculated value χ= 0,415 of the v. Kármán constant result and comply with the results above /11/.

3.3 Non-local mechanisms, the "fundamental paradox"

We discuss a new critical argument (2015), Ph. Spalart states that the "local" definition of the Navier-Stokes equations does not comply with the "non-local" character of turbulence. Spalart describes that interesting statement as "fundamental paradox" /14/. That corresponds to a "lack of amenability" to "single-point turbulence modelling", described by Mishra /15/. No consequences are discussed by both authors.

The author disagrees with Spalart's and Mishra's statements. If the experiment shows a "non-local" physical behaviour it is useless to try a "single point modelling" with a "local" set of equations. Consequence: The Navier-Stokes equations are simply the wrong equations for that physical problem. If they are the wrong equations, we will not find their solutions in the experiment.

The logarithmic "law-of-the-wall" is very successful for turbulence, but it is not perfect either. It fails completely due to the mathematical singularity near the wall. There the Navier-Stokes equations comply perfectly. Both solutions co-exist in the same experiment in different zones.

Fig. 6, Fig. 7 represent experimental and theoretical results depending of the "dimensionless wall distance" for the turbulent zone, the transition zone laminar-turbulent and the laminar near wall zone.

The interpolation curve in Fig. 6, Fig. 7 is based on the "logarithmic law-of-the-wall" and the Navier-Stokes solution – both crossing at $y^+ \approx 11$, remarkably out of the range of experimental points. It is evident that the logarithmic law fails near the wall and the laminar Navier-Stokes solution fails at higher wall distances.

The decision pro and contra the turbulent logarithmic law or the laminar Navier-Stokes solution requires the knowledge of the local lowest dissipation rate. That requires the knowledge of the local shear-stress. That important discussion is only possible with the Couette flow – we give it the separate next chapter.

3.4 Couette feature “constant shear stress”

The mentioned and following features are relevant for Spalart’s “fundamental paradox” and Mishra’s failing “single points turbulence modelling”.

The stationary Couette flow allows an interesting diagnosis which is not available for any other flow experiment. The shear stress is constant from wall to wall – equal to the wall shear stress - the profile being laminar, turbulent or both. It can be defined at any point of the profile, even at the turbulent zones. (E.g., the shear stress in the turbulent center of the pipe flow is not defined and unknown).

Prandtl made the assumption τ = const. in the boundary layer for the “mixing length hypothesis”, and the logarithmic law-of-the-wall. That is not correct but useful for the theory and disputed by Schlichting /1/ and Oertel jr. /17/. For Couette flow it is correct.

The Couette condition τ = const. and the Navier-Stokes definition of τ and τ_{ij} in Eq. 11, 11a requires for every Navier-Stokes solution

$$\frac{du}{dy} = \text{const.}$$

That means a constant gradient and consequently a linear Couette profile. Any S-shaped part of a turbulent Couette profile or part of a mixed turbulent-laminar Couette profile in Fig. 4 does not comply with a Navier-Stokes solution.

By the requirement of minimum dissipation different flow forms and their transition behaviour can be identified and explained. The dissipation rate is

$$\dot{E}(\mathrm{y}) = \tau_o \cdot \frac{du}{dy}\,(y) \tag{12}$$

proportional to wall shear stress τ_o and local gradient $\frac{du}{dy}$

The mentioned conditions $\tau = \text{const}.$ for the Couette flow means for any Navier-Stokes solution constant shear-stress, constant gradient and constant dissipation rate. The result is

- The logarithmic law is valid for the dimensionless wall distance $y^+ \geq 67{,}5$ (that corresponds to $\mathrm{Re} \geq 1750$)
- The Navier-Stokes equations are always valid below $y^+ < 2{,}5$

– The logarithmic law fails completely mathematically at low y^+

The most important statement is that in every "turbulent" Couette experiment there is always a non-turbulent zone near the wall, which can be described by the Navier-Stokes equations and can be considered as laminar. The thickness of this zone is by theory /11/ $y^+ < 2{,}5$, experimentally by Durst /10/ $y^+ < 2$, by Schlichting /1/ $y^+ < 5$.

We should always discuss this experiment as "mixed turbulent/laminar", the turbulent zone described theoretically by the logarithmic law above $y^+ > 67{,}5$ /11/, (experiment $y^+ > 60$ /1/, Fig. 6, Fig. 7.

The limits are defined theoretically /11/ by using only the Navier-Stokes equations, the logarithmic law of the wall, the minimum dissipation requirements, the condition τ = const. for Couette flow.

We state that at high Reynolds numbers there is always a Navier-Stokes zone near the wall within the mixed turbulent/laminar Couette experiment. The Navier-Stokes solution is unique and complies with the Couette condition τ = const., resulting due to equation 11,11a in a linear profile $\frac{du}{dy} = const.$

It is evident, that the logarithmic, S-shaped turbulent part of the profile (including the transition zone) has a lower dissipation rate than any linear/laminar part of the profile complying with τ = const. (see also Fig. 5).

The logarithmic law is a hypothesis, the Navier-Stokes equations are physical and mathematical theory. Like every physical theory both have their limits.

A transition laminar-to-turbulent in Fig. 4 is associated with an increase of the wall shear stress τ_o by a factor f $\geq$ 5,2 at Re = 1750 /1/. The transition at higher Re-numbers increases wall shear stress and dissipation rate by a factor f = 5-15 /1/, /11/. That is compensated by the S-shaped turbulent Couette profile with a local gradient, being reduced by that factor f in the center of the profile (see the arcs in Fig.4). The local dissipation rate at the turbulent center, being smaller than the dissipation rate of the former laminar profile, defines the Re-Number of the transition. That is quantified for the Couette flow in /11/.

The viscous sublayer of the profile near the wall is described by the Navier-Stokes Equations, complying with a lower local dissipation rate near the wall /11/.

We disagree with Spalart and Mishra. There is no “fundamental paradox” and “failing of single point modelling” with the Couette experiment. The zones in the centre of the mixed turbulent/laminar Couette experiment are not accessible to a Navier-Stokes solution. In the experimental they are simply not observed. On the other hand, the “very successful” logarithmic law fails completely near the wall – mathematically and due to the minimum dissipation requirement.

Such critical arguments are not new. In 1911 Lord Rayleigh (ex-student of Stokes in Cambridge, Nobel price 1904) mentions in a famous paper “The departure from Stokes law when the velocity is not very small” and summarizes that “Anyone who has looked over the side of a steamer will know that the motion is not usually of the kind supposed in the theory” /19/.

4. Summary

The “1. Stokes’ Problem” (the “suddenly accelerated flat wall”) is one of the oldest (1851) applications of the Navier Stokes Equations to instationary fluid flow situations. It is of special interest for turbulent flow that it takes into account extremely high shear stress, high local momentum exchange and high dissipation rates.

With the start Stokes’ solution includes extremely high shear rates near the “accelerated wall”: That contradicts the theorems of “Existence and Uniqueness” of partial differential equations (Cauchy 1841, Kowalewskaya 1875). That mathematical aspect is joined by a physical aspect of the thermodynamics of irreversible processes, the condition of minimum entropy production/ dissipation being violated. Extremely high local dissipation rates are mathematically and physically not possible.

The paper cites two excellent and well-known textbooks /1/, /2/. Both describe the same mathematical solution, but both drawings of the result (see Fig. 1 and Fig. 2) show a significant difference.

There exist other fluid flow mechanisms, co-existing with the Newton-Stokes “friction”. Examples are the kinetic theory of gases, other “corpuscular” models and the “universal” logarithmic “law-of-the-wall”, focussing non-laminar, turbulent mechanisms – which have to comply with the minimum dissipation requirement or are not observed in the experiment.

The “1. Stokes’ Problem“ has a second solution, the “suddenly accelerated Couette flow”. It has the described mathematical and physical problem with the jerk of the wall, but a “smooth” part of the solution valid for arbitrary long time intervals with a final linear, laminar stationary profile (the well-known laminar Couette profile). If that “smooth” part of the solution can be separated from the “difficult” zone near the wall it can be considered complying with the “Theorems of Existence and Uniqueness”. The direct consequence: the Navier Stokes solution is “unique” for a linear, laminar Couette profile – independent of the Re-number. That does not comply with the experiment. We do not need a final discussion of that aspect. If the laminar Stokes’ solution is unique, there is no turbulent solution. If not, there is no existence of any unique turbulent Navier-Stokes solution. The result is the same. The Navier-Stokes equations do not describe the turbulent part of the Couette experiment.

Critical remarks on the limited capability of the Navier-Stokes equations for turbulence have an interesting history.

Stokes himself focused on slow moving pendulums /18/ in a viscous fluid and mentions limitations to slow flow speeds.

O. Reynolds discussed "averaged" equations, named after him, thereby avoiding a lot of fundamental questions with turbulence, but creating the new "closure problem".

Lord Rayleigh (1911) mentions explicitly "infinitely slow" flow speeds. There is an important publication /19/ (cited by Schlichting /1/). Lord Rayleigh (ex-student of Stokes in Cambridge, Nobel price 1904) focusses "infinitely low" flow speeds and mentions the "departure from Stokes law when the velocity is not very small". His summary is brutal: "Anyone who has looked over the side of a steamer will know, that the motion is not usually of the kind supposed in the theory" /19/.

1963 came the excellent "Mathematical Theory of Viscous Incompressible Flow" /3/ of Olga Ladyzhenskaya. The AMS (American Mathematical Society) edited a complete journal on her life and work. The book became the specification of the "1 Mio. $ Millenium problem Navier-Stokes equations" of the Clay Mathematical institute in 2000. O. Ladyzhenskaya "recommends" to use the equations only at low Reynolds numbers and states: "It is hardly possible to explain the transition from laminar to turbulent flows within the framework of the classical Navier-Stokes theory". Stokes' Eq. (1) (under the name "heat transfer equation") is intensively discussed in the mathematical literature since > 150 years. Olga Ladyzhenskaya summarizes: "The basic problem of the unique solvability "in the large" of the boundary-value problem remains as open as ever" /3/. O. Ladyzhenskaya's "Mathematical Theory" /3/ includes remarkable and clear physical statements concerning turbulence. She discussed the Couette flow, but the Taylor-Couette flow between two concentric rotating cylinders, not the plane Couette flow, disputed here. She did not discuss the minimum dissipation requirement (Onsagers and Prigogines Nobel Prizes with that aspect were after her excellent book /3/).

A very interesting statement is < 10 years old. It is a thesis of non-compliance of the "non-local character" of turbulence with the "local" definition of the Navier-Stokes equations. Spalart /14/ defines a "Fundamental Paradox", Mishra /15/ a "lack of amenability" to "single point turbulence modelling".

The author was fascinated by the statements of Spalart and Mishra, but disagrees due to a special feature of the Couette experiment.

The Couette experiment is the only flow form with constant shear-stress $\tau = \tau_0 = const.$ from wall to wall, the flow being laminar, turbulent or both.

At high Reynolds numbers there is always a linear, laminar “viscous sublayer” near the wall, described perfectly by the Navier-Stokes equations and co-existing with a curved turbulent logarithmic flow profile at high wall distance. At that curved part of the flow profile the Navier-Stokes solution does not fail – it is simply not observed in the experiment, due to violation of the minimum dissipation requirement. On the other hand near the wall the turbulent logarithmic law (Prandtl’s mixing length hypothesis) fails completely due to a mathematical singularity and a higher dissipation rate compared to the Navier-Stokes solution.

Both, the logarithmic law and the Navier-Stokes equations have their limitations – like every physical theory. The minimum dissipation requirement and the condition constant shear-stress $\tau = \tau_0 = const.$ make straight consequences for the Couette flow.

Above $y^+ > 67{,}5$ turbulence is observed – the profile described by the logarithmic law – and no Navier-Stokes solution. Below $y^+ < 2{,}5$ a linear profile is observed – described by the Navier-Stokes equations – and the logarithmic law fails completely. The transition zone is observed – and not explained – but we can be sure that it complies with the minimum dissipation requirement and the condition $\tau = const.$

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