170 years “1. Stokes’ problem”,

contradicting experimental results

and mathematical deficits

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Abstract

The “1. Stokes’ problem (the “suddenly accelerated flat wall”) is the first non-stationary application of the Navier-Stokes Equations to a fluid experiment with extremely high, theoretically infinite, shear rates and corresponding local dissipation. A “Critical Review” states that a Navier-Stokes solution contradicts the “Theorem of Existence and Uniqueness of Partial Differential Equations” (Cauchy, Kowalewskaya) and the physical “Theorem of Minimum Entropy Production/Dissipation“ of the Thermodynamics of Irreversible Processes. The direct mathematical and physical consequence: There does not exist any correct Navier-Stokes solution, in spite of many historical and textbook articles and there is no physical experiment which verifies the flow profiles in the textbooks. The paper describes contradicting observations of a corresponding experiment. The results initiate a statement. The textbook solutions use mathematical methods which are not suitable for a qualified discussion of the above-mentioned consequences. There was a fundamental question. Can the Navier-Stokes’ Equation describe high shear fluid flow in general, e.g. turbulence? With regard to the consequences above they do not.

Keywords: 1. Stokes Problem, experimental results, mathematical deficits.

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1. Introduction

Stokes' early theory /1/ focuses on slow flow speeds. Lord Rayleigh in a famous paper on Stokes' subject mentions "infinitely slow motion" /2/.

Later authors (Ladyzhenskaya /3/, Fefferman /4/) require "smooth, physically reasonable solutions" for the Navier-Stokes equations and expect a "unique solution at least during a certain time interval" if the conditions are "not too bad from the standpoint of their smoothness". O. Ladyzhenskaya remarks on this relation: "The basic problem of the unique solvability 'in the large' remains as open as ever" /3/.

It is evident that the 1. Stokes problem in the detailed form of the "suddenly accelerated flat wall" or the "suddenly accelerated Couette flow" with extremely high shear rates, high shear stress and corresponding extremely high dissipation rates are not at all "smooth" and do not present "weak solutions".

The researcher, the engineer and the physicist will not limit his interest to "smooth" flow situations and "slow-motion" fluids. Lord Rayleigh criticized brutally in a famous paper: "Anyone who has looked over the side of a steamer will know that the motion is not usually of the kind supposed in the theory" /2/. J. Meixner is more polite: "Klassische Lehrbücher der theoretischen Physik behandeln das Continuum in der Regel ohne auf thermische Effekte einzugehen" /5/. (not translated intentionally).

High shear rates require sufficient input of energy, balanced by corresponding dissipation rate. These are Meixner's above mentioned "Thermische Effekte" (Meixner knew the "Minimum Dissipation Requirement"). The experiment "par excellence" is the "suddenly accelerated Couette flow". The "problem" is not the experiment, the problem is the theory.

The solution of the "1. Stokes problem" by the Navier-Stokes equations was not successful. Stokes himself – citing Cauchy in his famous paper /1/ - pioneered the substitution of a dimensionless variable in the partial Navier-Stokes differential equations, thereby converting these to an ordinary differential equation. Lord Rayleigh – 60 years later – confirmed Stokes' procedure /2/.

Stokes ignored the handicaps of ordinary differential equations. They are not equivalent to the partial Navier-Stokes differential equations. They are not unique due to an arbitrary number of solutions /6/ and there exists no "theorem of existence and uniqueness". This substitution of a dimensionless variable fixes the mathematical form of a (physical) result and makes the mathematical character of the solution useless as physical information.

2. The 1. Stokes' problem (textbook)

The "1. Stokes' problem" was described by Stokes in 1851. We follow the textbook description of Schlichting /6/. For the „suddenly accelerated" Couette flow, the partial differential equation is

$$\frac{\partial u}{\partial t} = \nu \cdot \frac{\partial^2 u}{\partial y^2} \qquad (1)$$

Identical to the heat transfer equation. By introducing the dimensionless variable η

$$\eta = \frac{y}{2\sqrt{\nu \cdot t}} \qquad (2)$$

and by setting U_0 = speed of the "jerk"

$$u = U_o \cdot f(\eta) \qquad (3)$$

results the ordinary differential equation

$$f'' + 2\eta \cdot f' = 0 \qquad (4)$$

By introducing a fixed wall opposite the accelerated wall at a distance h the calculation describes the sudden start of the Couette flow. The boundary conditions of the differential equations are $f = 1$ at the wall at $\eta = 0$ and $f = 0$ at $\eta = \eta_1 = \frac{h}{2}\sqrt{\nu \cdot t}$ defining the dimensionless distance of the two walls.

The application of the 1. Stokes' problem to a "sudden accelerated" Couette flow is described by Schlichting /6/ Fig. 1 shows the resulting flow profiles after the jerk. The final profile is the well-known stationary, linear, laminar Couette flow profile.

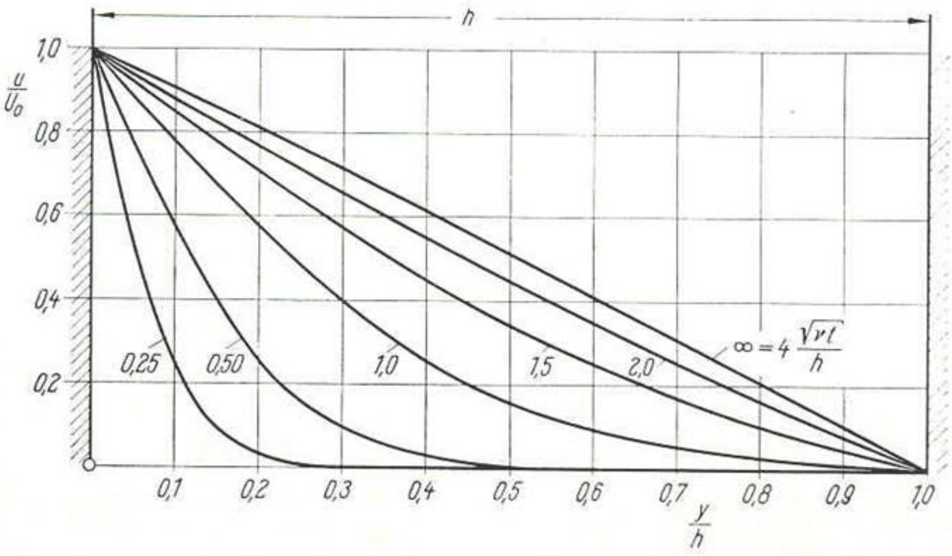


Fig. 1: "Jerky" acceleration of the Couette flow /6/

3. “Critical review” of the 1. Stokes’ solution

The “critical review” of the of the 1. Stokes’ solution by Drescher /8/ describes a non-mathematical, infinite gradient near the wall and a non-physical, infinite dissipation rate near the wall.

Eq. 1 and the boundary conditions contradict the theorem of “Existence and Uniqueness” of partial differential equations of Cauchy, Kowalewskaya /3/, /9/.

They further contradict the theorem of “minimum entropy production/dissipation” of the Thermodynamics of Irreversible Processes (Prigogine /10/, Klimontovich /11/).

The theorem of “Existence and Uniqueness” is known since 1875. The theorem of “minimum dissipation” is known since 1931 (two Nobel prices 1966, 1977).

The use of the mathematical and the physical knowledge of both theorems in the hydrodynamic literature has not been adequate in the past.

4. The jerk experiment

The 1. Stokes' problem identifies "non-smooth" flow situations, not accessible for the theory but observed in technology and experiment.

The focus of the jerk experiment is the "suddenly accelerated" Couette flow in form of the textbook solution of the 1. Stokes' problem /6/. The focus of the experiment is the difficult part of the "problem", the start with a "sudden jerk". Interesting are the first three to five seconds of the jerk. There is no suitable theory available.

The experiment was performed and documented in the Technicum of Bonnenberg & Drescher GmbH, Aldenhoven (www.budi.de). More than 30 experiments were performed in the last 12 months.

An "infinitely sudden" jerk in the experiment is not necessary for a significant result. To avoid later misunderstanding with the interpretation the experimental environment is chosen laminar. The Couette experiment consists of two opposite plates at a distance of 10 cm submerged in water. To perform the jerk a 3 kg metal weight is accelerated to a slow speed of 2 cm/s. One plate (200 g) of the Couette experiment is kicked by that weight. The Reynolds number is Re $\cong$ 1000 to avoid any discussion on transition to turbulence or hydrodynamic instability. The time of the movement is $\cong$ 3 sec.

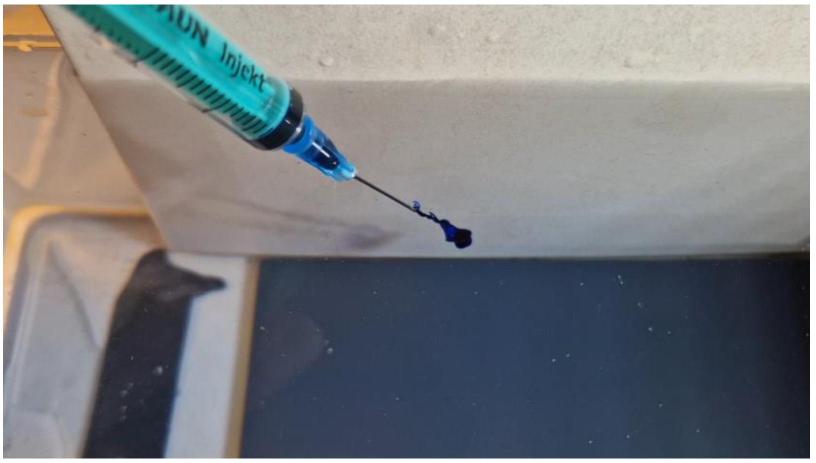
The fluid movement is made visible by slowly injecting ink at a position 2 to 20 mm off the kicked plate.

The parameters have to be varied in a range of (laminar) flow speeds. The speed of the jerk was varied at 1 to 2 cm/s, the distance of the Couette walls 5 to 15 cm. The motion of the jerk was limited to 4 cm.

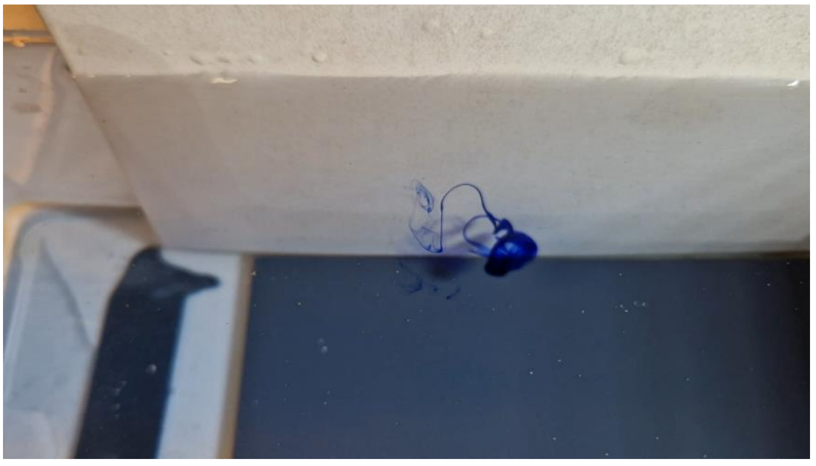
Visualizing the motion by ink is limited as ink cannot be injected very exactly. The position of the injected ink was varied between 2 mm and 20 mm off the wall.

None of the 30 experiments showed any similarity to the theoretical laminar results in Fig. 1.

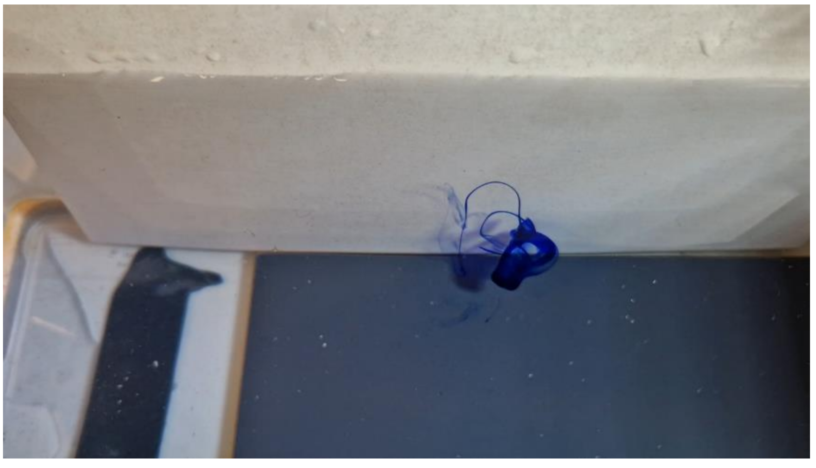
Fig. 2 shows a typical sequence of 0-3 seconds after the “jerk”. The focus of the experiment is a comparison with the theoretical textbook solution Fig. 1. The result of the experiment contradicts the textbook solution in Fig. 1 completely.



Jerk -2 seconds



Jerk +/-0 seconds



Jerk +1 second

Jerk +3 seconds, movement of plate finished

Jerk +6 seconds

Figure 2: The Jerk/Couette experiment

There is no mechanism (e.g. turbulent or other non-laminar) described in the literature, which is suitable to describe the observation. We state that the observation is not laminar as postulated by Fig. 1, but we are unable to specify its type (e.g. turbulent, transition forms etc.).

We cite again Lord Rayleigh: “The motion is not usually of the kind supposed in the theory” /2/.

5. The substitution of a variable – a persistent problem

The substitution of a variable η in Eq. 2 is pioneered by Stokes' - himself citing Cauchy in his famous publication 1851 /1/. The unknown function is u = u (y, t), substituted by the variable $\eta = \eta\left(\frac{y}{\sqrt{t}}\right)$.

The mathematical form of the unknown function u is of a different type than of the variable η fixed by that substitution. That is not acceptable for the solution near the wall and short times after the start of the "jerk". For great distances and long time-intervals the difference becomes significant and the result useless. We will discuss that for the "sudden start" of the Couette flow and the flat wall.

For the substituted ordinary differential equation Eq. (4) it is important that not only f (η), but any arbitrary function f (c + η) and f (c - η) is a solution. The mathematical procedure, described by Schlichting /6/, is not "unique".

The textbook solution of the "sudden start" of the Couette flow – dated 1960 – shows six different solution curves for different f (η) functions with different shape, finishing in the well-known linear laminar Couette flow profile "asymptotically" with $\sqrt{\nu \cdot t} \rightarrow \infty$ /6/.

Tab. 1 summarizes for every curve the consequences of the experimental parameters specified in Ch. 4. The distance between the two walls is 0,1 m, the medium is water (20 °C, $\nu = 10^{-6}$ m^2/s), the Reynolds number is Re = 1000.

Curve 6 is the final linear, laminar Couette flow profile. The first three curves are described "similar", near the accelerated wall. The remaining curves are influenced by the fixed wall at rest. This influence is described as "asymptotic approach" to the final stationary linear profile /6/.

The results for the required times t are surprising. The final profiles are reached after unrealistic long times. Schlichting /6/ mentions an "asymptotic approach" to the final well-known linear/laminar profile after a value t → ∞ (see Fig. 1)!

Such time values are nonsense. The result is published /6/ but not discussed.

Curve No.	$\frac{4\sqrt{\nu \cdot t}}{h}$ h = 0,1 m	t [s]
1	0,25	36
2	0,50	144
3	1,0	625
4	1,5	1300
5	2,0	2566
6	$\rightarrow \infty$	$\rightarrow \infty$

Tab. 1: Dimensional values in Fig. 1 according h = 0,1 m, Re = 1000

One focus of the “Critical review” /8/ is the theorem of “Existence and uniqueness” of the Navier-Stokes equations. It is evident that the substituted ordinary differential equations (Eq. 4) are not equivalent to the partial differential Navier-Stokes equation. The theorem of Cauchy, Kowalewskaya is only valid for partial differential equations. The substituted ordinary differential equations have an arbitrary number of different solutions.

We can summarize. The 1. Stokes’ problem with the basic equation (Eq. 1) has no suitable solution in the literature, the substituted Eq. (4) has no unique solution.

To discuss the situation near the accelerated wall we use the theoretical results for the “suddenly accelerated flat wall”. The boundary conditions for Eq. 4 are f = 1 near the wall for $\eta = 0$ and f = 0 for $\eta \rightarrow \infty$.

Compared to the Couette flow we cannot define a Reynolds number, but we can use the dimensional data of Ch. 4 to discuss a “laminar” environment.

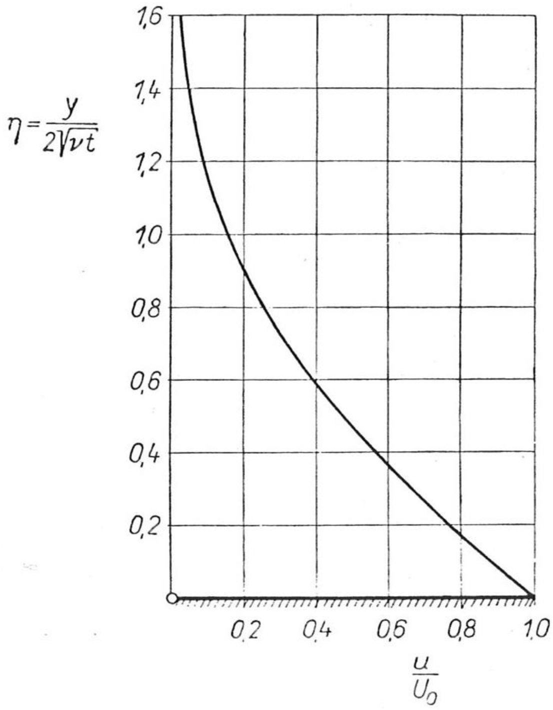


Fig. 3: Solution of the 1. Stokes' problem, Schlichting /6/

Fig. 3 shows the result near the suddenly accelerated flat wall /6/. The curve shows the function $f(\eta) = \frac{u}{U_0}$ with the x-axis and the y-axis with values of $\eta = \frac{y}{2\sqrt{\nu \cdot t}}$.

Tab. 2 summarizes the situation near the wall

η	$f(\eta) = \frac{u}{U_0}$
2,0	0,01
1,0	0,78
0,6	0,4
0,4	0,48
0,2	0,75
0,1	0,9
0	1

Tab. 2: Solution of Eq. 4, Fig. 3

It is evident that the curve in Fig. 3 representing $f(\eta) = \frac{u}{U_0}$ is “smooth”. The dimensionless gradient at the start of the “jerk” with values of $\frac{u}{U_0} \approx 0{,}8$ is

$$\frac{\delta f(\eta)}{\delta\,\eta} \approx \frac{(1{,}0-0{,}8)}{0{,}2} = \frac{0{,}2}{0{,}2} = 1 \qquad (5)$$

This has to be compared with the boundary condition of the “1. Stokes’ Problem” defining near the wall with y = 0 and t = 0 the start of the jerk

$$\frac{\delta u}{\delta y}(y) \to \infty$$

It is difficult to compare dimensionless parameters and results for extreme “non-smooth jerks”. But we can make use of the experiment in Ch. 4 by using the experimental parameters distance $y = 0{,}1$ m,

$$U_0 = 1\frac{cm}{s},$$

$$\nu = 10^{-6}\frac{m^2}{s}\ and$$

$$Re = 1000.$$

In Eq. (5) we replace $\delta\eta = \frac{y}{2\sqrt{\nu \cdot t}} = 0{,}2$, $\delta f(\eta) = 0{,}2$, $y = 0{,}1\,m$

resulting

$$\frac{\delta f(\eta)}{\delta\eta} = \frac{2 \cdot 10^{-3}\,m\sqrt{\frac{t}{s}}}{0{,}1\,m} = 1$$

$$= 2 \cdot 10^{-2} \cdot \sqrt{\frac{t}{s}}$$

$$\sqrt{\frac{t}{s}} = 50$$

$$t = 2500\,s$$

Such a time value t is nonsense.

Finally, we discuss a solution very near or at the wall. Very near the wall with small y-values we assume that the reduction of u and $\frac{u}{U_0}$ is a negative linear function of the wall distance y.

$$\delta\left(\frac{u}{U_0}\right) = \delta f = -const. \;\cdot y \tag{6}$$

$$\delta\eta = \frac{y}{2\sqrt{\nu \cdot t}}$$

$$\frac{\delta f}{\delta\eta} = -\frac{\frac{const.\cdot y}{y}}{2\sqrt{\nu\cdot t}}$$

$$= -\,const.\cdot 2\sqrt{\nu \cdot t}$$

$$= 0 \; for \; t \to 0$$

The solution of Eq. (6) does not and cannot describe a “non-smooth jerk” with local condition $\frac{du}{dy} \to \infty$.

An extremely high gradient is normally not the result of a function but is defined by “non-smooth” local boundary conditions or external forces, associated with corresponding high local energy input and dissipation rates.

The function $f(\eta)$ has no such definition of boundary conditions by corresponding external forces. The only boundary conditions near the wall are $f = 1 \; for \; \eta = 0$ (and $f = 0 \; for \; \eta = \infty$) resulting in a very “smooth” function of $f(\eta) = \frac{u}{U_0}$ (see Fig. 3).

The discussion of $\frac{du}{dy}\,(y)$ with $y \to 0$ is not precise. In the literature /2/, /7/ u is the unknown function in the Navier-Stokes equations and not dimensionless like the $f(\eta)$ function. dy is no independent and no dimensionless variable like $\delta\eta$. At the wall the definition of $f(\eta) = 1$ for $\eta = 0$ is the only discussed boundary condition.

For the function u there exist two definitions. The drawing of the “solution” Fig. 3 in Schlichting’s textbook /6/ shows for $\eta = 0$ varying values of u in the range

$$0 < u\,(0) < U_0$$

The drawing in Wieghardt’s textbook /7/ shows for $\eta = 0$ a fixed value of

$$u\,(0) = U_0$$

This is disputed in the “Critical review” /8/.

6. Remark on a thesis of Lord Rayleigh and Wieghardt

Both authors describe the substitution of the dimensionless variable η according Eq. (2) into the Navier-Stokes equations resulting in an ordinary differential equation according Eq. (4). For the result see Fig. 3. The formula for the solution is

$$u = U_0 - \frac{2\,U_0}{\sqrt{\pi}} \int_0^{\eta} e^{-\eta^2}\, d\eta \qquad /2/\ /6/\ /7/ \qquad (7)$$

Both authors are interested in the flow shear $\frac{du}{dy}$

$$\frac{du}{dy} = -\frac{U_0}{\sqrt{\pi \cdot \upsilon \cdot t}} e^{-\eta^2} \qquad /2/\ /7/ \qquad (8)$$

Their focus is on the wall shear stress. (Lord Rayleigh: "We require only the value of $\frac{du}{dy}$ when y = 0" /2/).

By simply cutting off the $e^{-\eta^2}$ function in Eq. (8) the consequence is wrong with

$$\frac{du}{dy\,|\,y=0} = -\frac{U_0}{\sqrt{\pi \cdot \upsilon \cdot t}} \quad \xrightarrow[t \to 0]{} \infty \qquad (8a)$$

Wieghardt uses Eq. (8a) also for calculation with y > 0.

At the wall both authors simply set y = 0. Both authors do not explain the shear stress going infinite with t → 0. Result: a physical surprise of a very primitive mathematical procedure.

At the most difficult point of the physical curve one cannot discuss a dimensionless function f (η) by simply truncating f (η) to $f\left(\frac{1}{\sqrt{t}}\right)$.

That discussion on extremely high ("infinite") local shear and consequently dissipation rates could have been qualified by introducing an early version of the "Minimum Dissipation Theorem" developed by Lord Rayleigh himself (and Helmholtz) and published by Wieghardt /7/.

The discussion can also be avoided by performing the differentiation of Eq. (7) correctly with $\frac{du}{d\eta}$ instead of $\frac{du}{dy}$ with the “smooth” result

$$\frac{du}{d\eta} = - \frac{2\,U_0}{\sqrt{\pi}} \cdot e^{-\eta^2} \tag{9}$$

The substituted ordinary differential equation Eq. (4) supplies a “smooth” solution, already well-known by Fig. 3.

The 1. Stokes’ Problem has an important and fascinating focus on high shear rates of a fluid. There is no problem with the experiment. The problem is with the theory.

7. Summary, consequences

- To apply his theory Stokes' 1851 requires "slow moving fluids" /1/. Lord Rayleigh requires "infinite slow motion" /2/. Newer authors Ladyzhenskaya /3/, Fefferman /4/ require "physically reasonable" solutions. They expect "uniqueness" if the conditions are "not to bad from the standpoint of their smoothness". The "1. Stokes' problem" does not comply with the required "smoothness" due to "infinite" shear rates.
- The "1. Stokes' problem" contradicts a solution by the Navier-Stokes equations due to the mathematical "Theorem of Existence and Uniqueness" (Cauchy, Kowalewskaya) and the physical theorem of "Minimum Entropy Production/Dissipation" of the Thermodynamics of Irreversible Processes (Onsager, Prigogine, Klimontovich)
- Contrary to this there are textbook publications resulting in extreme ("infinite") flow shear.
- There is a conflict between theory and experiment. The paper describes that 30 experiments with "suddenly accelerated Couette flow" have been made. No experiment showed any similarity to the textbook descriptions.
- Following the two theorems above the result is not surprising that there does not exist any Navier-Stokes solution on the "1. Stokes' Problem" in the literature. Beginning with Stokes' and Lord Rayleigh the Navier-Stokes partial differential equations are substituted by a dimensionless ordinary differential equation, which introduces a different mathematical character of the solution.
- First consequence: That "solution" is not unique.
- Further consequence: The result of extremely high ("infinite") shear near the wall at $t = 0$, described 1911 by Lord Rayleigh and published 62 years later by Wieghardt is wrong. The correct mathematical solution gives a very "smooth" result (see Ch. 6).
- Further result: The Navier-Stokes solutions do not describe high flow shear. Due to the minimum dissipation requirement, we expect limitations with e.g. turbulent shear flow with significant shear and dissipation rates.
- For the Couette experiment the minimum dissipation requirement explains and characterizes the co-existence of laminar and not laminar flow zones, the co-existence of laminar and turbulent zones, the local limitations of the different zones and three different types of transition laminar-turbulent-laminar. The value of the empirical v. Kármán constant is calculated /13/ /14/ /15/.

- The mathematical theorem (existence, uniqueness) and the physical theorem (minimum dissipation requirement) support significant consequences. Their focus is different.

  The mathematical theorem is only valid for partial differential equations. There may be other mathematical theories that may be useful in the same field, which are perhaps more simple and need not be unique.

  The physical theorem has no relation to any mathematical theory. It has a high ranking in thermodynamics and is present everywhere in our physical environment. If a theoretical result contradicts the "requirement" it is wrong, if a theoretical result ignores it, the result is at least incomplete.

  Example: The laminar-linear Couette experiment and the laminar Hagen-Poiseuille flow in the tube experiment is perfectly described by the Navier-Stokes equations. As the solution is "smooth" and complies with the theorem of Cauchy- Kovalevskaya the solution is unique. At higher Reynolds number we observe a turbulent or a mixed turbulent-laminar flow with a flatter profile. That means that the turbulent part of the mixed turbulent-laminar profile cannot be described by a unique solution of the Navier-Stokes equations.

8. Acknowledgement

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