Calculation of load carrying capacity of shell structures with elasto-plastic material by direct methods

Thanh Ngoc Tran[1], M. Staat[1], R. Kreißig[2]

[1]Aachen University of Applied Sciences, Campus Jülich, Ginsterweg 1, 52428 Jülich, Germany
[2]Chemnitz University of Technology, Faculty of Mechanical Engineering, Straße der Nationen 62, 09107 Chemnitz, Germany

In this paper, a method is introduced to determine the limit load of general shells using the finite element method. The method is based on an upper bound limit and shakedown analysis with elastic-perfectly plastic material model. A non-linear constrained optimisation problem is solved by using Newton's method in conjunction with a penalty method and the Lagrangean dual method. Numerical investigation of a pipe bend subjected to bending moments proves the effectiveness of the algorithm.

1. Introduction

In practical engineering, the calculation of the load carrying capacity for structures has been a problem of great interest to many designers. In the early 20th century, it could be relatively easily obtained by imposing the stress intensity at a certain point of the structure equal to the yield stress of the material. This implies that structural failure occurs before yielding. However, many materials, for example the majority of metals, exhibit distinct, plastic properties. Such materials can deform considerably without breaking, even after the stress intensity attains the yield stress. This implies that if the stress intensity reaches the critical (yield) value, the structure does not necessarily fail or deform extensively. To this case, in order to permit higher loads, elastic-plastic structural analysis becomes more general than the classical elastic one. Among the plasticity methods, Limit and Shakedown Analysis (LISA) seems to be the most powerful one. In Europe LISA have been developed as direct plasticity methods for the design and the safety analysis of severely loaded engineering structures, such as nuclear power plants and chemical plants, offshore structures etc. Staat (2002; Staat and Heitzer, 2003). Annex B of the new European pressure vessel standard EN 13445-3 is based on LISA (European standard, 2005-06), (Taylor et al., 1999) thus indicating the industrial need for LISA software. All design codes are based on perfect plastic models. The extension of LISA to hardening materials is no problem (Staat and Heitzer, 2002).

Shell structures are used in many engineering applications due to efficient load carrying capacity relative to material volume. From the engineering point of view, shells often allow to build structures with high strength and stiffness and relatively low weight. From the analysis point of view, shell structures stand for a challenging problem mostly due to the three-dimensional finite rotations. Plasticity in shell structures is accounted either by means of integration of stress over the thickness (layer approach), or stress resultant modelling. By the latter approach the yield surface becomes more complicated than with the former approach, thus requiring a special consideration in algorithm for updating of the stress resultant.

This paper concerns the application of a kinematic formulation for the finite element limit and shakedown analysis of general thin shells. The technique is based on an upper bound approach using the re-parameterized exact Ilyushin yield surface and a non-linear optimization procedure. The solution of the problem is obtained by discretizing the shell into finite elements. It is typical for the direct plasticity methods that the development of algorithms for the structural problem is influenced by the material modelling.

2. Plastic dissipation function in term of stress resultants

Let h be the shell thickness and σ_y the uniaxial yield stress. The non-dimensional 'engineering' stress and strain resultant vectors are introduced as follows

$$\begin{aligned} &\tilde{\boldsymbol{\sigma}}=\begin{bmatrix}\mathbf{n} & \mathbf{m}\end{bmatrix}^T,\ \mathbf{n}=\frac{1}{N_0}\begin{bmatrix}N_{11} & N_{22} & N_{12}\end{bmatrix}^T,\ \mathbf{m}=\frac{1}{M_0}\begin{bmatrix}M_{11} & M_{22} & M_{12}\end{bmatrix}^T \\ &\tilde{\boldsymbol{\varepsilon}}=\begin{bmatrix}\overline{\mathbf{e}} & \mathbf{k}\end{bmatrix}^T,\ \overline{\mathbf{e}}=\frac{1}{\varepsilon_0}\begin{bmatrix}\overline{\varepsilon}_{11} & \overline{\varepsilon}_{22} & 2\overline{\varepsilon}_{12}\end{bmatrix}^T,\ \mathbf{k}=\frac{1}{\kappa_0}\begin{bmatrix}\kappa_{11} & \kappa_{22} & 2\kappa_{12}\end{bmatrix}^T \end{aligned} \tag{1}$$

in which $N_0=\sigma_0 h$, $M_0=\sigma_0 h^2/4$, $\varepsilon_0=\sigma_0(1-\nu^2)/E$ and $\kappa_0=4\varepsilon_0/h$ are the normalized quantities. The quadratic strain intensities are defined in terms of the incremental 'engineering' strain resultant by

$$\begin{aligned} &P_\varepsilon=\frac{3}{4}\left(d\overline{\mathbf{e}}^p\right)^T\mathbf{P}^{-1}d\overline{\mathbf{e}}^p=\frac{3}{4}\left(d\tilde{\boldsymbol{\varepsilon}}^p\right)^T\mathbf{P}_1 d\tilde{\boldsymbol{\varepsilon}}^p \qquad (\geq 0) \\ &P_{\varepsilon\kappa}=3\left(d\overline{\mathbf{e}}^p\right)^T\mathbf{P}^{-1}d\mathbf{k}^p=3\left(d\tilde{\boldsymbol{\varepsilon}}^p\right)^T\mathbf{P}_2 d\tilde{\boldsymbol{\varepsilon}}^p \\ &P_\kappa=12\left(d\mathbf{k}^p\right)^T\mathbf{P}^{-1}d\mathbf{k}^p=12\left(d\tilde{\boldsymbol{\varepsilon}}^p\right)^T\mathbf{P}_3 d\tilde{\boldsymbol{\varepsilon}}^p \qquad (\geq 0) \end{aligned} \tag{2}$$

where $\mathbf{P}$ and its inverse $\mathbf{P}^{-1}$, $\mathbf{P}_i$ $(i=1,2,3)$ are

$$\begin{aligned} &\mathbf{P}=\begin{pmatrix}1 & -1/2 & 0\\ -1/2 & 1 & 0\\ 0 & 0 & 3\end{pmatrix},\ \mathbf{P}^{-1}=\begin{pmatrix}4/3 & 2/3 & 0\\ 2/3 & 4/3 & 0\\ 0 & 0 & 1/3\end{pmatrix}, \\ &\mathbf{P}_1=\begin{pmatrix}\mathbf{P}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix},\ \mathbf{P}_2=\begin{pmatrix}\mathbf{0} & \mathbf{P}^{-1}/2\\ \mathbf{P}^{-1}/2 & \mathbf{0}\end{pmatrix},\ \mathbf{P}_3=\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{P}^{-1}\end{pmatrix}. \end{aligned} \tag{3}$$

Ilyushin (1948) published the derivation of a stress resultant yield surface, describing the case where a cross-section of a shell is fully plastified. However, this yield surface has not been used because the parametric form in which it was described was not amenable to calculation. In order to avoid the difficulties arising with the parameterization of the Ilyushin yield surface and to use the exact yield surface in practical computations, Burgoyne and Brennan (1993) introduced the parameters

$$\upsilon=\frac{P_\varepsilon}{P_\kappa},\quad \beta=-\frac{P_{\varepsilon\kappa}}{P_\kappa}\quad \text{and}\quad \gamma=\upsilon-\beta^2 \tag{4}$$

With these parameters, the plastic dissipation function for a shell structure may be written in the form (Tran et al, 2007)

$$D^p(\dot{\tilde{\boldsymbol{\varepsilon}}}^p)=N_0\varepsilon_0\sqrt{\frac{P_\kappa}{3}}\left(\beta_1\sqrt{\beta_1^2+\gamma}+\beta_2\sqrt{\beta_2^2+\gamma}+\gamma K_0\right) \tag{5}$$

where β_1, β_2 and K_0 are

$$\begin{aligned} &\beta_1=0.5-\beta,\quad \beta_2=0.5+\beta \\ &K_0=\ln\left(\frac{\sqrt{(0.5-\beta)^2+\gamma}+(0.5-\beta)}{\sqrt{(0.5+\beta)^2+\gamma}-(0.5+\beta)}\right) \end{aligned} \tag{6}$$

It is to be noted that D^p is convex but not everywhere differentiable (Capsoni and Corradi, 1997). In order to allow a direct non-linear, non-smooth, constrained optimization problem, a "smooth regularization method" should be used by adding to γ and P_κ a small positive number, namely η^2. Thus, in this case, equation (6) is amenable to a numerical evaluation for all values of $\dot{\tilde{\boldsymbol{\varepsilon}}}^p$.

3. Upper bound limit and shakedown algorithm for general shell structures

Consider a convex polyhedral load domain $\mathcal{L}$ and a special loading path consisting of all load vertices $\hat{P}_k$ ($k = 1,...,m$) of $\mathcal{L}$. By discretizing the whole structure by finite elements and application of Koiter's theorem, the shakedown limit, which is the smaller one of the low cycle fatigue limit, and the ratchetting limit may be found by the following minimization

$$\alpha^+ = \min \sum_{k=1}^{m} \sum_{i=1}^{NG} w_i N_0 \varepsilon_0 \sqrt{\frac{P_\kappa}{3}} \left(\beta_1 \sqrt{\beta_1^2 + \gamma} + \beta_2 \sqrt{\beta_2^2 + \gamma} + \gamma K_0 \right)$$

$$\text{s.t.:} \quad \begin{cases} \sum_{k=1}^{m} \dot{\tilde{\boldsymbol{\varepsilon}}}_{ik} = \mathbf{B}_i \dot{\mathbf{u}} & \forall i = 1,..., NG \\ \sum_{k=1}^{m} \sum_{i=1}^{NG} w_i N_0 \varepsilon_0 \dot{\tilde{\boldsymbol{\varepsilon}}}_{ik}^T \tilde{\boldsymbol{\sigma}}_{ik}^E = 1 \end{cases} \tag{7}$$

in which $\mathbf{B}_i$ denotes the deformation matrix, $\dot{\mathbf{u}}$ is the displacement rate vector, w_i is the weighting factor of the Gauss point i^{th} and NG is the total number of Gauss points in the structure. By introducing some new notations

$$\dot{\mathbf{e}}_{ik} = w_i \dot{\tilde{\boldsymbol{\varepsilon}}}_{ik}, \quad \mathbf{t}_{ik} = N_0 \varepsilon_0 \tilde{\boldsymbol{\sigma}}_{ik}^E, \quad \hat{\mathbf{B}}_i = w_i \mathbf{B}_i \tag{8}$$

where $\dot{\mathbf{e}}_{ik}, \mathbf{t}_{ik}, \hat{\mathbf{B}}_i$ are the new strain rate vector, new fictitious elastic stress vector, and new deformation matrix, respectively, we obtain a simplified version for the upper bound shakedown analysis

$$\alpha^+ = \min \sum_{k=1}^{m} \sum_{i=1}^{NG} N_0 \varepsilon_0 \sqrt{\frac{P_\kappa}{3}} \left(\beta_1 \sqrt{\beta_1^2 + \gamma} + \beta_2 \sqrt{\beta_2^2 + \gamma} + \gamma K_0 \right)$$

$$\text{s.t.:} \quad \begin{cases} \sum_{k=1}^{m} \dot{\mathbf{e}}_{ik} = \hat{\mathbf{B}}_i \dot{\mathbf{u}} & \forall i = 1,..., NG \\ \sum_{k=1}^{m} \sum_{i=1}^{NG} \dot{\mathbf{e}}_{ik}^T \mathbf{t}_{ik} = 1. \end{cases} \tag{9}$$

By applying Newton's method in conjunction with a penalty method and the Lagrangean dual method to solve the KKT conditions of system (9) we obtain the Newton directions $d\dot{\mathbf{u}}$ and $d\dot{\mathbf{e}}_{ik}$, which assure that a suitable step along them will lead to a decrease of the objective function α^+. If the relative improvement between two steps is smaller than a given constant, the algorithm stops and leads to the shakedown limit factor (Tran et al., 2007). It is noted, that if there is only one load and this load does not vary, then the load domain reduces to a point ($k = 1$). This fact means that the above upper bound of shakedown load factor reduces to that of limit load factor.

4. Numerical example

Consider an 90^o elbow with mean radius r, bend radius of curvature R and thickness h. One of its ends is supposed clamped and the other one is subjected to a constant in-plane closing moment M_I or a constant out-of-plane bending moment M_{II} as shown in figure 1b. The curvature factor is defined as follow

$$\lambda = \frac{Rh}{r^2} \tag{10}$$

Generally, $\lambda \leq 0.5$ corresponds to a highly-curved pipe, while $\lambda \to \infty$ corresponds to a straight pipe. In order to evaluate the model, different values of λ within the range $[0.1,\ 1.2]$ are examined. Our model that is used for elastic-plastic analysis is meshed by 700 quadrangular flat 4-node shell elements as shown in figure 1a. The elastic-perfectly plastic material model is used with $E = 208000$ MPa , $\nu = 0.3$, $\sigma_y = 250$ MPa .

Elbow under in-plane closing bending moment

We define the limit load factor $\alpha_I = M_I / M_I^s$, where M_I is the limit moment of the elbow and M_I^s is the limit moment of the straight pipe which has the same radius as the elbow. Calladine (1974) proposed a lower bound solution for an infinite, strongly-curved pipe ($\lambda \le 0.5$)

$$\alpha_I^C = 0.9346\lambda^{2/3}. \tag{11}$$

This solution is considered in the literature to come close to the experimental limit load factor (Bolt and Greenstreet, 1972; Goodall, 1978; Griffiths, 1979). According to Yan (1997), it is a good approximation when $\lambda < 0.7$. For a slightly-curved pipe ($\lambda \ge 0.7$), he proposed an approximate solution which is validated by numerical analysis

$$\alpha_I^Y = \cos(\frac{\pi}{6\lambda}). \tag{12}$$

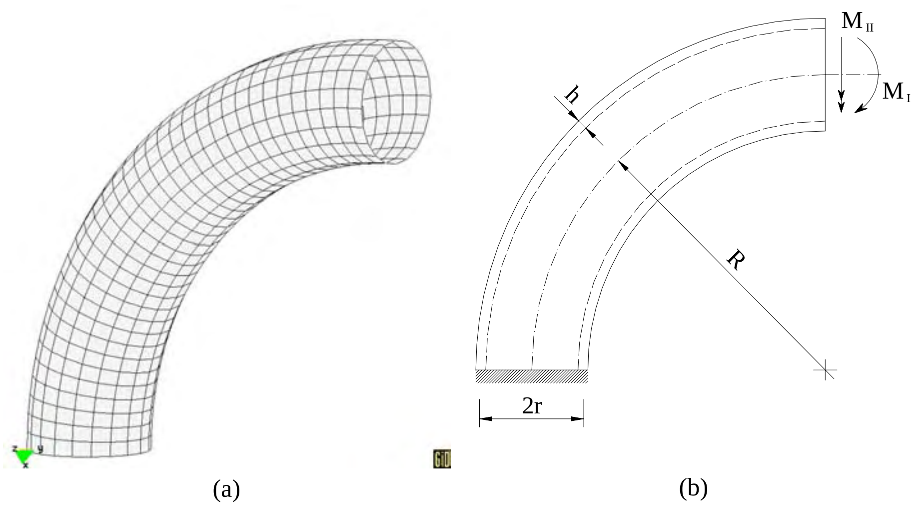

Figure 1. FE-mesh and geometrical dimensions

In the framework of his PhD thesis, Desquines et al. (1997) proposed a more general analytical solution as a lower bound, which can applied for any value of λ

$$\alpha_I^{De} = \frac{1}{\sqrt{1+\frac{0.3015}{\lambda^2}}}. \tag{13}$$

Spence and Findlay (1973) also expressed an analytical solution for the limit load of an elbow

$$\alpha_I^{SF} = 0.8\lambda^{0.6}, \quad \lambda < 1.45. \tag{14}$$

All the foregoing expressions are based on small displacement analysis and assume perfectly plastic material behavior. Based on large displacement analysis, Goodall (1978) proposed the maximum load-carrying capacity of the elbow subjected to closing bending moment as

$$\alpha_I^G = \frac{1.04\lambda^{2/3}}{1+\beta}. \tag{15}$$

where

$$\beta = \frac{4\sqrt{3(1-\nu^2)}r\sigma_y}{\pi Eh}\left[2+\frac{(3\lambda)^{2/3}}{3}\right]$$

Table 1: Limit load factors of elbow under in-plane closing bending moment

λ	Calladine (11)	Yan (12)	Desquines (13)	Spence & Findlay (14)	Goodall (15)	Touboul (16)	Drubay (17)	Present paper
0.100	0.2013	-	0.1791	0.2001	0.1489	0.154	0.1657	0.2155
0.200	0.3196	-	0.3422	0.3046	0.2817	0.2445	0.263	0.3279
0.250	0.3709	-	0.4144	0.3482	0.3401	0.2838	0.3052	0.3900
0.300	0.4188	-	0.4794	0.3885	0.3947	0.3204	0.3446	0.4614
0.363	0.4756	-	0.5515	0.4355	0.4593	0.3638	0.3913	0.5200
0.400	0.5074	-	0.5888	0.4617	0.4955	0.3882	0.4175	0.5589
0.500	0.5887	-	0.6732	0.5278	0.5879	0.4504	0.4844	0.6260
0.600	0.6648	-	0.7377	0.5888	0.6741	0.5087	0.5471	0.6930
0.650	0.7013	-	0.7639	0.6178	0.7153	0.5365	0.577	0.7227
0.700	0.7368	0.7330	0.7868	0.6458	0.7554	0.5637	0.6063	0.7494
0.750	-	0.7660	0.8069	0.6732	0.7945	0.5902	0.6348	0.7773
0.800	-	0.7933	0.8244	0.6998	0.8327	0.6162	0.6627	0.7974
0.903	-	0.8365	0.8544	0.7525	0.909	0.668	0.7184	0.8341
1.000	-	0.8660	0.8765	0.8000	0.9782	0.715	0.769	0.8617
1.200	-	0.9063	0.9093	0.8925	1.1138	0.8074	0.8684	0.9006

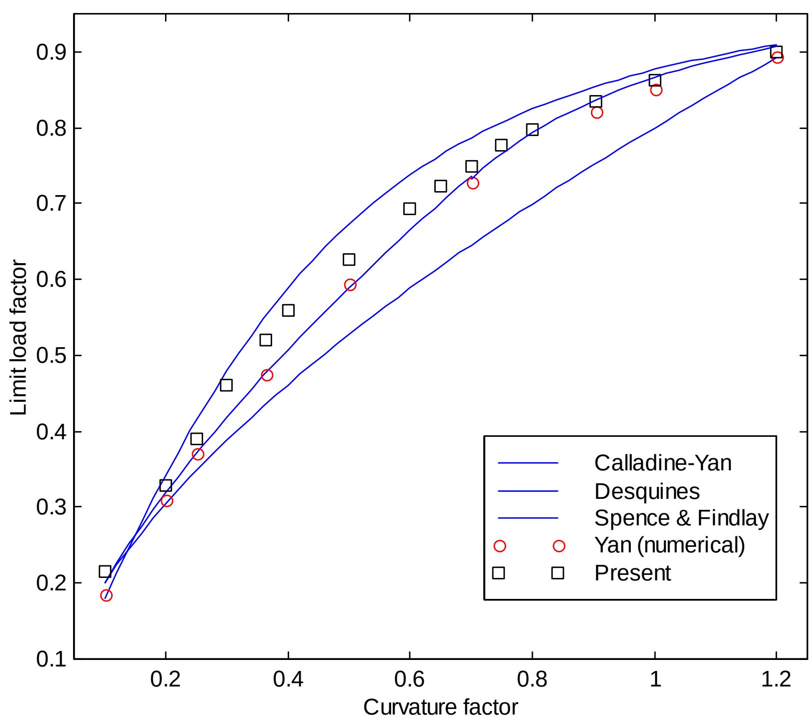

Figure 2. Limit load factors of elbow under in-plane closing bending moment

Table 2: Limit load factors of elbow under out-of-plane bending moment

λ	Yan (18a)	Yan (18b)	Present paper
0.100	0.2763	-	0.2476
0.200	0.4188	-	0.4047
0.250	0.4788	-	0.4709
0.300	0.5341	-	0.5244
0.363	0.5989	-	0.5675
0.400	0.6348	-	0.6063
0.450	0.6813	-	0.6337
0.500	0.7257	0.7143	0.6575
0.550	-	0.7374	0.6924
0.600	-	0.7591	0.7245
0.650	-	0.7796	0.7539
0.700	-	0.7991	0.7808
0.750	-	0.8177	0.8053
0.800	-	0.8355	0.8276
0.903	-	0.8699	0.8674
1.000	-	0.9000	0.8984
1.200	-	0.9564	0.9467

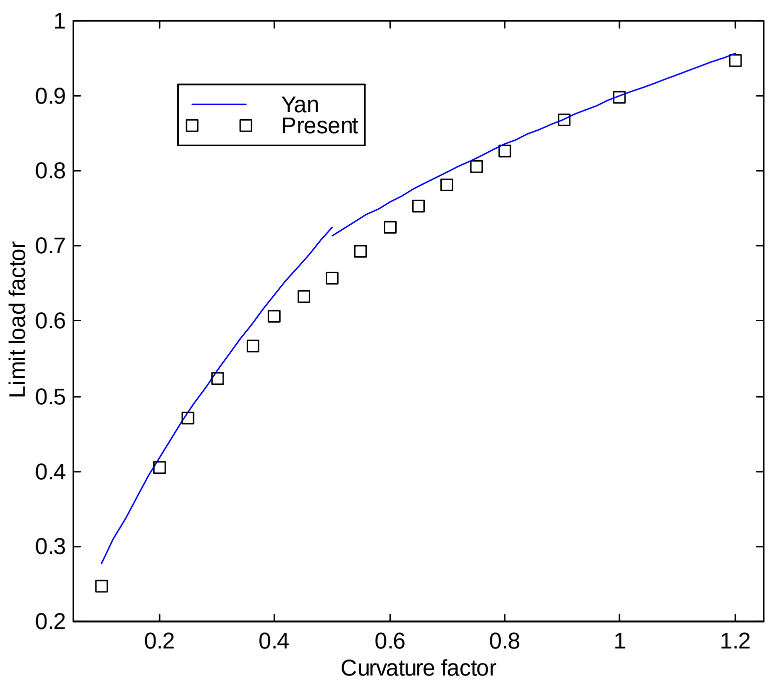

Figure 3. Limit load factors of elbow under out-of-plane bending moment

Based on the experimental study at CEA DEMT, Touboul et al. (1989) proposed the following equations of closing collapse moments of elbows

$$\alpha_I^T = 0.715\lambda^{2/3} . \tag{16}$$

Drubay et al. (1995) expressed another closing mode collapse moments of elbows as

$$\alpha_I^{Dr} = 0.769\lambda^{2/3} . \tag{17}$$

Our numerical results are introduced in table 1 and figure 2, compared with these above analytical solutions and a numerical solution of Yan (1997). It is shown that our solutions compare well with the other analytical solutions, which are based on small displacement theory, but bigger than those which are based on large displacement theory. They converge as an upper bound of Calladine's solution and lower bound of Desquines's solution.

Elbow under out-of-plane bending moment

We define the limit load factor $\alpha_{II} = M_{II} / M_{II}^t$, where M_{II} is out-of-plane limit moment of the elbow, M_{II}^t is the torsion limit moment of the axle which has the same radius as the elbow
By this definition, Yan (1997) proposed an analytical solution for the elbow subjected to out-of-plane bending moment

$$\alpha_{II}^Y = 1.1\lambda^{0.6}, \qquad \lambda < 0.5 , \tag{18a}$$

$$\alpha_{II}^Y = 0.9\lambda^{1/3}, \qquad 0.5 \le \lambda \le 1.4 . \tag{18b}$$

Numerical results are introduced in table 2 and figure 3, compared with the analytical solution of Yan (1997). It is shown that our solutions compare well with Yan's solution outside the range $0.4 \le \lambda \le 0.7$.

5. Conclusions

The numerical solutions demonstrate that the proposed method is capable of identifying reasonable estimates of the limit load factor for a wide range of thin shell problems. It has been tested against several limit loads which have been calculated in literature. A numerically very effective method is achieved from the lesser computational cost by using shell elements compared with volume elements and by direct plasticity methods which achieve plastic solutions in the computing time of only several linear elastic steps. This method seems to be particularly suited to comparatively large problems or to the application in structural optimization and structural reliability.

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