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UNCERTAIN MULTIMODE FAILURE AND LIMIT ANALYSIS OF SHELLS

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Abstract. This paper presents a numerical procedure for reliability analysis of thin plates and shells with respect to plastic collapse. The procedure involves a deterministic limit analysis for each probabilistic iteration, which is based on the upper bound approach and the use of the exact Ilyushin yield surface. Probabilistic limit analysis deals with uncertainties of the loads, material strength and thickness of the shell. Based on a direct definition of the limit state function, the calculation of the failure probability may be efficiently solved by using the First and Second Order Reliability Methods (FORM and SORM). The problem of reliability of structural systems (series systems) will be handled by the application of a special technique which permits to find all the design points corresponding to all the failure modes. Studies show, in this case, that it improves considerably the FORM and SORM results.

1 INTRODUCTION

The reliability analysis of plates and shells with respect to plastic collapse or to inadaptation was formulated on the basis of limit and shakedown theorems [1]. The technique was based upon an upper bound approach using a re-parameterized exact Ilyushin yield surface and nonlinear optimization procedures. Based on a direct definition of the limit state function, the non-linear problems may be efficiently solved by using the First and Second Order Reliability Methods (FORM and SORM). In order to get the design point, a non-linear optimization was implemented. FORM and SORM match particularly well with direct plasticity methods because they render the problem time invariant and they calculate sensitivities effectively from quantities already obtained by the minimization of the upper bound.

The non-linear optimization algorithm developed in [1] is guaranteed to converge to a minimum-distance point on the limit state surface, provided that the limit state function is continuous and differentiable. However, as with any non-convex optimization problem, it is not guaranteed that the solution point will be the global minimum-distance point when the system has more than one failure mode. This paper aims at extending the method developed in [1] for the probabilistic shakedown analysis of multimode-failure of plate and shell structures. A method to successively finding the multiple design points of a component reliability problem, when they exist on the limit state surface, is presented. Each design point corresponds with an individual failure mode or mechanism. FORM and SORM approximations are applied at each design point followed by a series system reliability analysis to lead to improved estimates of the system probability of failure.

2 PROBABILISTIC SHAKEDOWN ANALYSIS OF SHELLS

Let $\mathbf{X} = (X_1, X_2, ..., X_n)$ be an *n*-dimensional random variable vector characterizing uncertainties in loads, material strength and shell thickness, and $\mathbf{x} = (x_1, x_2, ..., x_n)$ be *realizations* of $\mathbf{X}$. The deterministic safety margin is based on the comparison of a structural resistance (threshold) R and loading S. With R, S are functions of $\mathbf{X}$, the structure fails for any realization with non-positive *failure function* or *limit state function*, i.e.

$$g(\mathbf{X}) = R(\mathbf{X}) - S(\mathbf{X}) \begin{cases} < 0 & \text{for failure,} \\ = 0 & \text{for limit state,} \\ > 0 & \text{for safe structure.} \end{cases} \quad (1)$$

The limit state function $g(\mathbf{x}) = 0$, defines the limit state hyper-surface ∂F which separates the failure region $F = \{\mathbf{x} | g(\mathbf{x}) < 0\}$ from the safe region. The failure probability P_f is the probability that $g(\mathbf{X})$ is non-positive, i.e.

$$P_f = P(g(\mathbf{X}) \le 0) = \int_F f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}, \quad (2)$$

where $f_{\mathbf{X}}(\mathbf{x})$ is the *n*-dimensional joint probability-density function. In general, evaluation of the integral in (2) for an arbitrary failure region may not be possible. Therefore, approximation methods are obviously needed. In this study, the First- and Second-Order Reliability Methods (FORM and SORM) are used to evaluate the failure probability. FORM and SORM are the most effective methods if gradient information is available [2].

2.1 Definition of the limit state function

As mentioned above, the limit state function contains the parameters of structural resistance and loading. By normalization with the actual load P, the limit state function can be expressed by the form [1, 3]

$$g = \alpha_{\lim} - 1, \quad (3)$$

in which $\alpha_{\lim}$ is the shakedown load factor calculated from an optimization problem. Details of a numerical algorithm for shakedown analysis of thin shells can be found in [4]. The limit state function is the function of the yield stress variable X_{n-1}, thickness variable X_n and load variables X_j ($j = 1,...,n-2$). The actual load *Q*, in general, can be defined by its components as follows

$$Q = \sum_{j=1}^{n-2} x_j Q_j^0, \tag{4}$$

where x_j and Q_j^0 are the realization and constant reference load of the j^{th} basic load variable X_j, respectively. By that way, the actual stress resultants $\boldsymbol{\sigma}$ can also be decomposed as

$$\boldsymbol{\sigma} = \sum_{j=1}^{n-2} x_j \boldsymbol{\sigma}_j^0. \tag{5}$$

The Jacobian and the Hessian of the limit state function, $\partial g / \partial \mathbf{x}$ and $\partial^2 g / \partial \mathbf{x}^2$, repectively are needed for the FORM and SORM as well as for finding the most likely failure points. They must be first calculated at each probabilistic iteration in the physical **x**-space. Then it is transferred into the standardized Gaussian **u**-space by using the chain rule

$$\nabla_u g(\mathbf{u}) = \nabla_u g(\mathbf{x}) = \nabla_x g(\mathbf{x}) \nabla_u \mathbf{x}, \tag{6}$$

$$\nabla_u^2 g(\mathbf{u}) = \nabla_u \left(\nabla_x g(\mathbf{x}) \nabla_u \mathbf{x} \right) = \left(\nabla_u \mathbf{x} \right)^T \nabla_x^2 g(\mathbf{x}) \nabla_u \mathbf{x} + \nabla_x g(\mathbf{x}) \nabla_u^2 \mathbf{x}.$$

The calculation of the Jacobian and the Hessian in the physical **x**-space is based on a sensitivity analysis. Details of the calculation can be found in [3].

2.2 First- and Second-Order Reliability Methods

It is welknown that the most essential contributions to the failure probability come from the vicinity of the most likely failure point if the distance from the origin in the standardized Gaussian space to this point is suitably large [5]. The most likely failure point or the design point is the point on the limit state surface that has the shortest distance to the origin in the **u**-space. FORM and SORM are analytical probability integration methods in which the limit state function are approximated by a linear or second-order surface at the design point in the **u**-space. If the limit state function is not strictly non-linear, the probability of failure P_f can be determined with good accuracy by FORM as

$$P_{f,I} = \Phi(-\beta_{HL}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\beta_{HL}} e^{-0.5z^2} dz, \tag{7}$$

where β_{HL} is the distance from the origin in the **u**-space to the design point on the limit state surface and $\Phi(.)$ is the standard Gaussian distribution function. β_{HL} is defined as the shortest distance from origin to the limit state surface ∂F, i.e.

$$\beta_{HL} = \min_{g(\mathbf{u})=0} \sqrt{\mathbf{u}^T \mathbf{u}}. \tag{8}$$

A non-linear optimization algorithm which is based on the sequential quadratic programming (SQP) is adopted to solve the optimization problem in (8). Details of the algorithm can be found in [1].

As an attempt to improve the accuracy of FORM, the limit state surface is approximated by a quadratic hypersurface with the main curvatures κ_j ($j = 1,...,n-1$) at the design point are equal to those of the limit state surface in SORM. The failure probability is then calculated as a three term approximation [6]

$$P_{f,II} = S_1 + S_2 + S_3, \tag{9}$$

with

$$S_1 = \Phi(-\beta_{HL}) \prod_{j=1}^{n-1} \left(1 - \beta_{HL}\kappa_j\right)^{-1/2},$$

$$S_2 = \left[\beta_{HL}\Phi(-\beta_{HL}) - \phi(\beta_{HL})\right] \left\{ \prod_{j=1}^{n-1} \left(1 - \beta_{HL}\kappa_j\right)^{-1/2} - \prod_{j=1}^{n-1} \left(1 - (\beta_{HL}+1)\kappa_j\right)^{-1/2} \right\},$$

$$S_3 = (\beta_{HL}+1)\left[\beta_{HL}\Phi(-\beta_{HL}) - \phi(\beta_{HL})\right] \left\{ \prod_{j=1}^{n-1} \left(1 - \beta_{HL}\kappa_j\right)^{-1/2} - \mathrm{Re}\left[\prod_{j=1}^{n-1} \left(1 - (\beta_{HL}+i)\kappa_j\right)^{-1/2} \right] \right\}$$

where $i = \sqrt{-1}$, $\mathrm{Re}[.]$ represents the real part of the complex argument and $\phi(.)$ is the standard Gaussian probability density function.

3 MULTIMODE FAILURE

Structural systems can generally be characterized as series or parallel systems or some combination of the two [7]. In series system, the formation of any individual failure mode or mechanism is defined as system failure. For example, in statically determinate or rigid-plastic structures, formation of a collapse mechanism will result in failure of the total system and therefore they can be modelled as series system with each element of the series being a failure mechanism. In parallel system, failure in a single element will not result in failure of the system, because the remaining elements may be able to sustain the external loads by redistributing of the loads. A typical example of a parallel system is a statically indeterminate structure. Failure of such structures will always require that more than one element fails before the structure loses integrity and fails.

System-reliability analysis concerns the calculation of the failure probability when the structure has more than one failure mode (multimode failure). Mathematically we encounter system reliability analysis if the limit state surface is composed by more pieces that generally intersect pairwise in sets of singular points. Each of these pieces corresponds with an individual failure mode or mechanism. This section aims at presenting a method to successively find all the design points of a system-reliability problem, if they exist on the limit state surface. FORM and SORM approximations are applied at each design point followed by a series system reliability analysis to lead to improved estimates of the system failure probability.

3.1 Bounds for the system probability of failure

If there are q failure mechanisms and the limit state surface is respectively described by q equations

$$g_i(\mathbf{X}) = g_i(X_1,...,X_n) = 0, \quad i = 1,...,q \tag{10}$$

and if we denote the failure due to the i^{th} mode as the random event $E_i = \{\mathbf{x} \mid g_i(\mathbf{X}) \le 0\}$, then the probability that the system fails is the probability that any failure mechanism occurs. It means that

$$P_f = P\left(E_1 \cup E_2 \cup ... \cup E_q\right) = P\left(\bigcup_{i=1}^{q} E_i\right). \tag{11}$$

If the joint probability density function of the failure events $f_E(\mathbf{e})$ is known, then the system probability of failure can be calculated by the q-dimensional integral

$$P_f = P\left(\bigcup_{i=1}^{q} E_i\right) = \int_{-\infty}^{0} ... \int_{-\infty}^{0} f_E(e_1,...,e_q)de_1...de_q. \tag{12}$$

Generally, evaluation of the system probability of failure through direct integration of (12) may not be feasible, even if an expression exists for the joint density function of failure modes and all failure modes have been identified. In this case, bounds relieve the necessity of evaluating the q-dimensional integral either analytically, numerically or through Monte Carlo simulation with some variance reduction [8], [9]. Several first-order bounds exist (e.g. [10]) which only require knowledge of the individual probabilities of failure resulting directly from the axioms of the probability theory. Unfortunately, these bounds may be quite wide for structural reliability application [11]. Closer or second-order bounds can be given in terms of the individual failure probabilities and the joint failure probabilities between any two modes. If we denote the individual failure probabilities as

$$P_i = P\left[g_i(\mathbf{X}) \le 0\right], \ \ i = 1,...,q, \tag{13}$$

then the bounds of the system probability of failure for a series system are [12]

$$P_1 + \sum_{i=2}^{q} \max\left\{P_i - \sum_{j=1}^{i-1} P_{ij}, \ 0\right\} \le P_f \le \sum_{i=1}^{q} P_i - \sum_{i=2}^{q} \max_{j<i}\left(P_{ij}\right), \tag{14}$$

where the notation P_{ij} has been used for the joint failure probability

$$P_{ij} = P\left[g_i(\mathbf{X}) \le 0, \ g_j(\mathbf{X}) \le 0\right]. \tag{15}$$

Since not all couples of the random events E_i are taken into account in equation (14) the ordering of the modes will have an effect on the bounds. Practical experience suggested that ordering the failure modes according to decreasing values P_i may correspond to the better bounds. In structural reliability, these bounds are frequently used and are considered sufficiently accurate for most structural systems [11].

3.2 First-order system reliability analysis

In a first-order system reliability analysis, the failure set is approximated by the polyhedral set bounded by the tangent hyper-planes at the design points. Each design point corresponds to a failure mode and they are the points on the limit state surface that have smallest distances to the origin in the **u**-space. We denote the design points in the **u**-space as $\mathbf{u}_i^*,\ i = 1, 2, ..., q$ and associated with each design point, we define the distance $\beta_{HLi} = \left\| \mathbf{u}_i^* \right\|$, which is the corresponding reliability index. The individual probabilities of failure P_i are determined as

$$P_i = \Phi(-\beta_{HLi}). \tag{16}$$

The first-order approximation to P_{ij} is obtained by approximating the joint failure set by the set bounded by the tangent hyper-planes at the design points for the two failure modes. The joint failure probability P_{ij} is thus calculated as

$$P_{ij} = \Phi\left(-\beta_{HLi}, -\beta_{HLj}; \rho_{ij}\right) = \int_{-\infty}^{-\beta_{HLi}} \int_{-\infty}^{-\beta_{HLj}} \varphi\left(x, y; \rho_{ij}\right) dx dy, \tag{17}$$

where the correlation coefficients between two failure modes ρ_{ij} are

$$\rho_{ij} = \cos \nu_{ij} = \frac{\left(\mathbf{u}_i^*\right)^T \mathbf{u}_j^*}{\beta_{HLi} \beta_{HLj}}, \tag{18}$$

and $\varphi\left(x, y; \rho\right)$,

$$\varphi\left(x, y; \rho\right) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2} \frac{x^2 + y^2 - 2\rho x y}{1-\rho^2} \right], \tag{19}$$

is the probability density function for a bivariate normal vector with zero mean values, unit variances and correlation coefficient ρ. Substituting the density function in (17) by the corresponding cumulative distribution function $\Phi\left(x, y; \rho\right)$, which gives

$$\begin{aligned} P_{ij} &= \Phi\left(-\beta_{HLi}, -\beta_{HLj}; 0\right) + \int_0^{\rho_{ij}} \frac{\partial \Phi\left(-\beta_{HLi}, -\beta_{HLj}; z\right)}{\partial \rho} dz \\ &= \Phi\left(-\beta_{HLi}\right) \Phi\left(-\beta_{HLj}\right) + \int_0^{\rho_{ij}} \varphi\left(-\beta_{HLi}, -\beta_{HLj}; z\right) dz. \end{aligned} \tag{20}$$

Numerical techniques are available for evaluating the joint failure probability in equation (20), e.g. Newton-Codes method. Simple bounds on the joint failure probability, which is based on geometrical illustration of a multimode failure system, can also be given, thus avoiding any numerical integration [7]. It should be noted that the bounds (14) still estimate a solution of the generally unknown region which respect to the exact value of the probability of failure. If we do not know where the values of the probabilities are placed with respect to the exact values, we cannot confirm that the bounds given above estimate the probability of failure. They bound some approximation and we can only more-or-less reasonably expect that the approximation is close to the exact result and the bounds remain meaningful.

3.3 Calculation of the multiple design points

A real structure may have several failure modes or failure mechanisms and the existence of multiple failure modes (or multiple design points) may cause the following problems in FORM and SORM. That is, the optimization algorithm which was developed in [1] may converge to a local design point. In that case, the FORM/SORM solution will miss the region of dominant contribution to the failure probability integral and, thus, has large error. Even if the global design point is found, the neighborhoods of the local design points may also have a significant contribution to the failure probability integral. Approximating the limit state surface only at the global design point will lose these contributions.

In this section, a simple method is presented for finding the multiple design points of a system reliability analysis problem, when they exist on the limit state surface. The method was developed by Der Kiureghian and Dakessian [13]. The basic idea of the method is to construct "barriers" around previously found solutions, thus forcing the algorithm to seek a new solution. Once all the design points are known, the failure probability of series system is calculated by using first-order system reliability method and second-order bounds as presented above.

Suppose that the first design point $\mathbf{u}_1^*$ is already found by the non-linear optimization algorithm developed in [1]. A "barrier" for this point is then constructed by adding a 'bulge' to the limit state surface. Thus, the limit state function for the deformed surface is

$$g_1(\mathbf{u}) = g(\mathbf{u}) + B_1(\mathbf{u}), \tag{21}$$

where $B_1(\mathbf{u})$ defines the bulge fitted at $\mathbf{u}_1^*$. Solving the optimization problem with the new limit state function $g_1(\mathbf{u})$ leads to a second design point $\mathbf{u}_2^*$. In order to seek the third solution point $\mathbf{u}_3^*$, a bulge $B_2(\mathbf{u})$ is now added at $\mathbf{u}_2^*$ resulting in the new limit state function $g_2(\mathbf{u}) = g_1(\mathbf{u}) + B_2(\mathbf{u})$. The process is repeated until all design points are found. The limit state function for finding the q^{th} design point thus, is

$$g_{q-1}(\mathbf{u}) = g(\mathbf{u}) + \sum_{i=1}^{q-1} B_i(\mathbf{u}). \tag{22}$$

Details of the definition of the bulges $B_i(\mathbf{u})$ can be found in [13]. As is shown in Fig. 1, it is possible for the optimization algorithm to converge to the points located at the feet of the bulge, which are actually the spurious minimum-distance points. However, practical experience showed that this occurs only when there is no other genuine design point. Thus, convergence to a spurious point usually means that no other genuine design point exists [13]. This nature can be used as the stopping criterion of the algorithm.

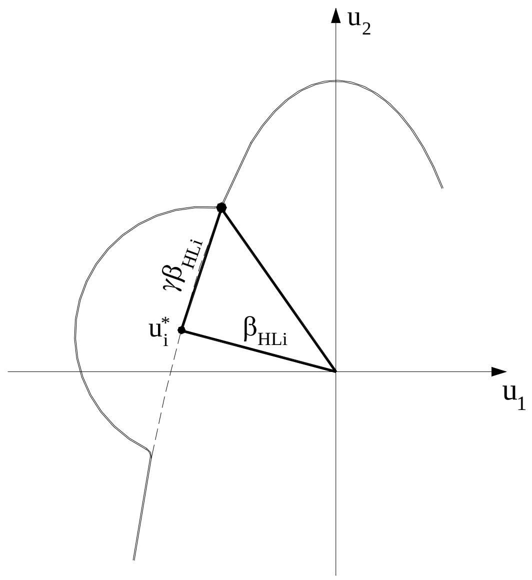

Figure 1: Definition of a bulge at design point $\mathbf{u}_i^*$

4 NUMERICAL EXAMPLE

A numerical example is presented in this section to assess the performance of the proposed method. In this example, a well-known problem with several failure modes is investigated. Consider the frame formed of three plates in Fig. 2 (left) which is generated by extruding a plane frame in the third direction. It is subjected to a constant uniform horizontal and a constant vertical load *H* and *V*. The loading and geometrical data were selected to match those of the plane frame included in the book of Madsen et al. [7]. Loads and limit plastic bending moment (material strength) are random variables which are assumed mutually independent and log-normally distributed. There are three basic variables and their mean values and standard deviations are given in table 1.

The thickness of the shell is supposed to be constant $h=0.4\,\mathrm{m}$. The plastic moment capacity M_p at each section is then a random variable with mean value and standard deviation

$$\mu_{M_p} = \frac{bh^2}{4}\mu_{\sigma_y} = 134.9 \text{ kNm}, \qquad (23)$$

$$\sigma_{M_p} = \frac{bh^2}{4}\sigma_{\sigma_y} = 13.49 \text{ kNm}.$$

Hinge lines are thought to form at the end of elements (beam and columns) or at lines of load application. As in the original plane frame, three failure modes caused by plastic hinge mechanisms are expected to occur. Those are sway mode, frame mode and beam mode (Fig. 3).

Table 1: Mean values and standard deviations of basic variables - Folding shell

	Horizontal load H (kN/m)	Vertical load V (kN/m)	Yield stress σ_y (kN/m^2)
Mean value μ	50	40	3372.5
Standard deviation σ	15	12	337.25

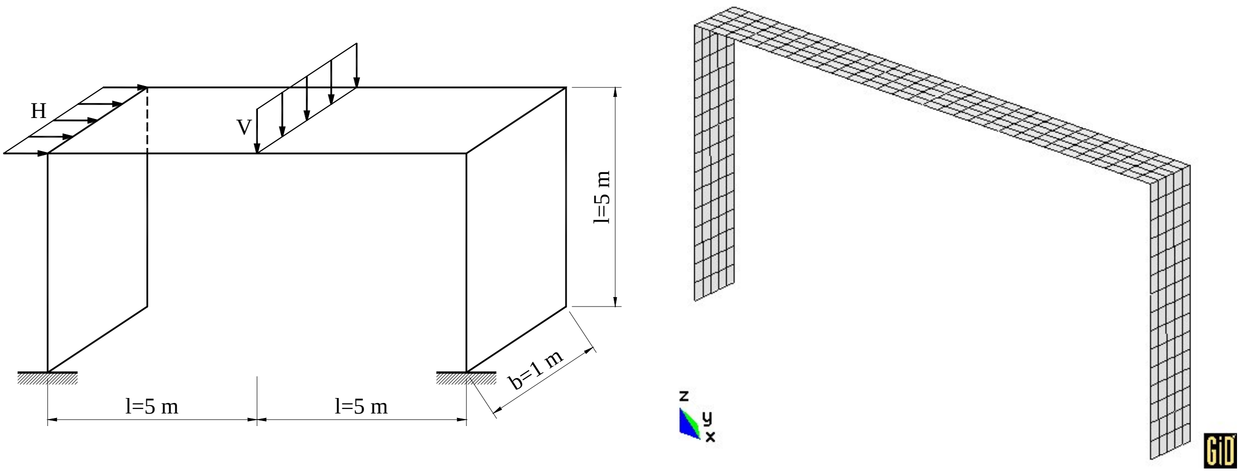

Figure 2: Geometrical dimensions and FE-mesh of the frame

Numerical computation is carried out by using 300 quadrangular flat 4-node shell elements as shown in Fig. 2 (right). The 'barriers' technique developed by Der Kiureghian and Dakessian is performed with $\gamma = 0.4$, $\delta = 0.3$ in order to find all the three design points as expected. γ and δ are two parameters used to define the bulges (see [13]). Our numerical results of design points are presented in table 2. The global design point $\mathbf{u}_1^* = 3.083[0.944 \quad 0.024 \quad -0.329]^T$ with $\beta_{HL1} = 3.083$ was found firstly in the $\mathbf{u}$-space. Three components of $\mathbf{u}$ are the sensitivities of the three basic variables: *H*, *V* and yield stress σ_y, respectively. This design point corresponds to the sway mode since the sensitivieties show that the effect of the horizontal load H is dominant. After adding a bulge $B_1(\mathbf{u})$ at $\mathbf{u}_1^*$, the algorithm converges to the second design point $\mathbf{u}_2^* = 3.24[0.776 \quad 0.47 \quad -0.422]^T$ with $\beta_{HL2} = 3.240$. This design point corresponds clearly to the frame mode because both of the loads have similar sensitivities and so both have large contributions to the failure probability of the structure. We continuously added a bulge $B_2(\mathbf{u})$ at $\mathbf{u}_2^*$ and found the third design point $\mathbf{u}_3^* = 3.461[0.457 \quad 0.783 \quad -0.421]^T$ with $\beta_{HL3} = 3.461$ which corresponds to the beam mode. Now we suppose to proceed further and place a bulge $B_3(\mathbf{u})$ at $\mathbf{u}_3^*$. Our search algorithm now converges to $\mathbf{u}_4^* = 3.307[0.925 \quad -0.325 \quad -0.194]^T$ with $\beta_{HL4} = 3.307$. The distance $\|\mathbf{u}_4^* - \mathbf{u}_1^*\| = 1.218$ between the two design points is less than but close to the radius $r_1 = 0.4 \times 3.083 = 1.233$ of the bulge, thus confirming that $\mathbf{u}_4^*$ is a spurious design point. If we further place a bulge $B_4(\mathbf{u})$ at $\mathbf{u}_4^*$ and continue the algorithm, the point $\mathbf{u}_5^* = 4.07[0.667 \quad 0.59 \quad -0.454]^T$ with $\beta_{HL5} = 4.070$ is found. Obviously it is also a spurious design point since the distances $\|\mathbf{u}_5^* - \mathbf{u}_3^*\| = 1.24$ between the two design points $\mathbf{u}_3^*$, $\mathbf{u}_5^*$ and $\|\mathbf{u}_5^* - \mathbf{u}_2^*\| = 1.028$ between the two design points $\mathbf{u}_2^*$, $\mathbf{u}_5^*$ are less than the radius $r_3 = 0.4 \times 3.461 = 1.384$ and $r_2 = 0.4 \times 3.24 = 1.296$ of the bulges. Thus, at this stage, we stop to search and assume that there are only three design points for this problem. It should be noted here that, the SQP algorithm worked well in this case to seek all the optimal points.

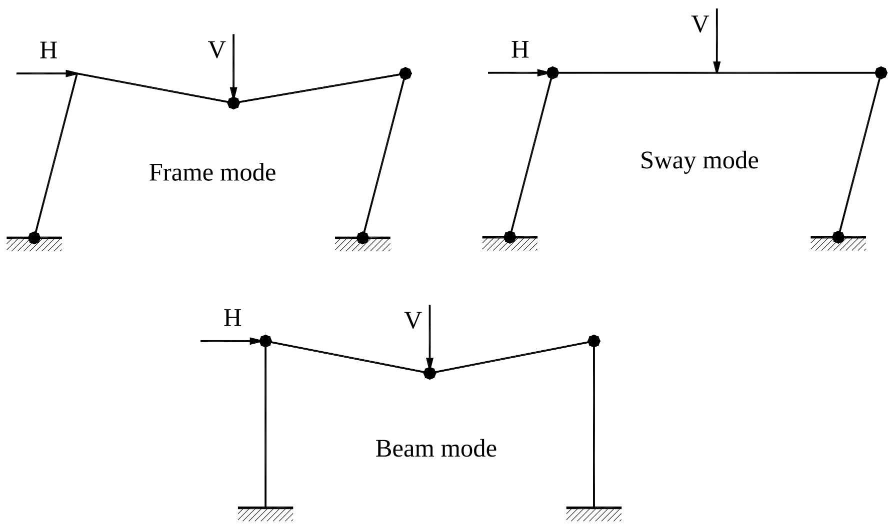

Figure 3: Three failure modes of the frame

Table 2: Multiple design points and search steps - Folding shell ($a_i = u_i / \|\mathbf{u}\|$)

Step	a_1	a_2	a_3	β_{HL}	Design point	Mode
1	0.944	0.024	– 0.329	3.083	global	Sway mode
2	0.776	0.470	– 0.422	3.240	local	Frame mode
3	0.457	0.783	– 0.421	3.461	local	Beam mode
4	0.925	– 0.325	– 0.194	3.307	spurious	-
5	0.667	0.590	– 0.454	4.070	spurious	-

Table 3: Failure mode correlations and joint failure mode probabilities - Folding shell (first order)

	Failure mode correlations ρ_{ij}			Joint failure probabilities $P_{ij} \times 10^2$		
	1	2	3	1	2	3
1	1.0	0.883	0.589	0.1023	0.0301	0.0037
2	0.883	1.0	0.899	0.0301	0.0598	0.0155
3	0.589	0.899	1.0	0.0037	0.0155	0.0269

Table 4: Failure probability of the folding shell **(** $P_f \times 10^2$ **)**

Method	$\mathbf{u}_1^*$ alone (sway)		$\mathbf{u}_2^*$ alone (frame)		$\mathbf{u}_3^*$ alone (beam)		$\mathbf{u}_1^*$, $\mathbf{u}_2^*$ and $\mathbf{u}_3^*$
	FORM	SORM	FORM	SORM	FORM	SORM	FORM
Present	0.1023	0.1026	0.0598	0.110	0.0269	0.0166	0.140-0.143
Madsen et al. [9]	0.336	0.322	0.199	0.267	0.0291	0.0283	0.467

Table 5: Reliability indices of the folding shell β_{HL}

Method	$\mathbf{u}_1^*$ alone (sway)		$\mathbf{u}_2^*$ alone (frame)		$\mathbf{u}_3^*$ alone (beam)		$\mathbf{u}_1^*$, $\mathbf{u}_2^*$ and $\mathbf{u}_3^*$
	FORM	SORM	FORM	SORM	FORM	SORM	FORM
Present	3.083	3.082	3.240	3.062	3.461	3.589	2.982-2.989
Madsen et al. [9]	2.710	2.725	2.880	2.786	3.440	3.447	2.600

The linear approximation of the failure set is now constructed by the tangent hyper-planes at the three design points. The corresponding approximations of failure mode correlations and joint failure mode probabilities are listed in table 3. The single and system failure probabilities and reliability indices are presented in table 4 and in table 5, respectively, compared with those of the original plane frame obtained by Madsen et al. [7]. It is shown that our two first failure probabilities of sway and frame modes are smaller than those of plane frame while the last one compares well with the solution of Madsen et al. leading to a smaller failure probability of the system. It is understandable since the stress state is now three-dimensional, not only the bending moment but also the compression force contribute to the failure of the structure.

The shell problem of a pipe T-junction with the two failure modes burst of the large pipe and fully plastic bending of the small pipe is discussed with generalization to failure by inadaptation in [14] and shows FORM solutions which are close to the analytical failure probabilities. For space limitations, the full detail of probabilistic limit and shakedown analysis with FORM/SORM for general structures could not be presented here but it is given in [15] for a single failure mode.

5 CONCLUSIONS

The present work provides a direct plastic analysis method for the integrity assessment of shell structures with multimode failure. Practical experience showed that the existence of multiple design points in component reliability analysis could give rise to large errors in FORM and SORM approximations of the failure probability. In this work, a technique has been applied with a SQP algorithm to successively find the multiple design points of a system reliability problem, if they exist on the limit state surface. This technique is based upon a 'barrier' method by constructing a bulge around previously found design points, thus forcing the algorithm to seek a new one. Second-order bounds of the reliability of series system are then calculated based on the first-order system reliability analysis. The application of the method is demonstrated with a simple numerical example.

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