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DIRECT FEM LIMIT AND SHAKEDOWN ANALYSIS WITH UNCERTAIN DATA

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Abstract. *The structural reliability with respect to plastic collapse or to inadaptation is formulated on the basis of the lower bound limit and shakedown theorems. A direct definition of the limit state function is achieved which permits the use of the highly effective first order reliability methods (FORM) is achieved. The theorems are implemented into a general purpose FEM program in a way capable of large-scale analysis. The limit state function and its gradient are obtained from a mathematical optimization problem. This direct approach reduces considerably the necessary knowledge of uncertain technological input data, the computing time, and the numerical error, leading to highly effective and precise reliability analyses.*

1 INTRODUCTION

Present structural reliability analysis is typically based on the limit state of initial or local failure. This may be defined by first yield or by some member failure if the structure can be designed on an element basis. However, this gives quite pessimistic reliability estimates, because virtually all structures are redundant or statically undetermined. They are quite safe beyond initial failure. Progressive member failures of such systems reduce redundancy until finally the statically determined system fails. This system approach is not defined in an obvious way for a finite element (FE) representation of a structure.

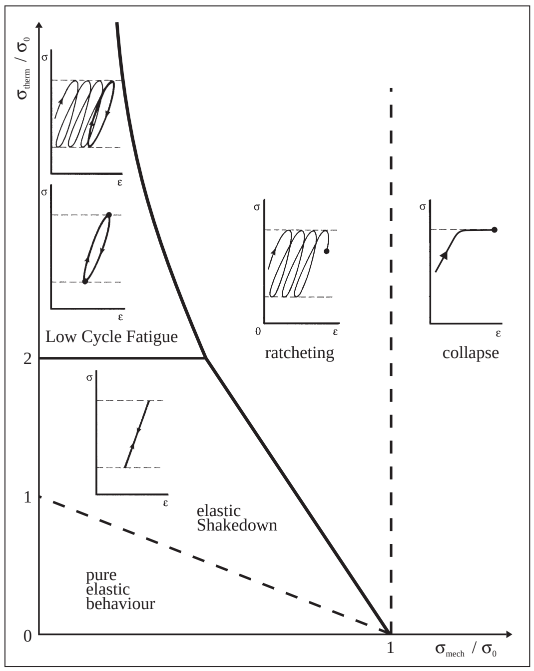


Figure 1: Bree-Diagram of thin wall tube

Collapse is the loss of stiffness. Ratchetting and low cycle fatigue (LCF) must be read from time evolution of plastic deformation as sketched in the Bree-Diagram[3], Fig. 1. Both, loss of stiffness and time evolution are difficult to use in a mathematical expression of the limit state function separating failure from safe structure. For loss of stiffness the smallest eigenvalue of a large FEM stiffness matrix should be monitored, but this is numerically difficult and expensive. Moreover, the reliability problem becomes a non-linear first passage problem. Until today first order reliability methods (FORM) could not be used with standard incremental plastic analysis because non-linear sensitivity analysis would be necessary for computing the gradient of the limit state function. Therefore one was restricted to simple but ineffective Monte-Carlo Simulation (MCS) and mostly local failure definitions.

All these problems are overcome by direct limit and shakedown analyses, because they compute directly the load carrying capacity or the safety margin. Therefore, they may be used to combine finite element methods (FEM) with FORM for defining the failure. Moreover, the solution of the resulting optimization problem provides the sensitivities with no extra costs. In comparison with MCS a speed up of some 100 and 1000 is achieved with limit and shakedown analysis, respectively. The direct approach computes safety without going through the different progression of local failures for all possible load histories. Moreover, it needs only key strength information of the material. Therefore, limit and shakedown analysis is an obvious choice for reliability analysis of structural problems with uncertain data of loading and of the structure.

2 RELIABILITY ANALYSIS

The behaviour of a structure is influenced by various typically uncertain parameters (loading type, loading magnitude, dimensions, material data,...). All parameters are described by random variables collected in the vector of basis-variables $\mathbf{X} = (X_1, X_2, ...)$. We will restrict ourselves to those basis–variables X_j which could be described by densities f_j, such that the joint density $f(x_1, \ldots, x_n)$ exists and the joint distribution function $F(\mathbf{x})$ is given by

$$F(\mathbf{x}) = P(X_1 < x_1, \ldots, X_n < x_n) \tag{1}$$

$$= \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f(t_1, \ldots, t_n) dt_1 \ldots dt_n. \tag{2}$$

The deterministic safety margin $R - S$ is based on the comparison of a structural resistance R and loading S (which is usually a local stress at a hot spot or in a representative cross-section). With $R,\ S$ function of $\mathbf{X}$ the structure fails for any realization with non-positive limit state function

$$g(\mathbf{X}) = R - S. \tag{3}$$

The limit state $g(\mathbf{X}) = 0$ defines the limit state surface ∂V which separates the failure region V from the safe region. The failure probability P_f is the probability that $g(\mathbf{X})$ is non-positive, i.e.

$$P_f = P(g(\mathbf{X}) \leq 0) = \int_V f_X(\mathbf{x}) d\mathbf{x}. \tag{4}$$

Usually, it is not possible to calculate P_f analytically. The computational effort of straight forward MCS increases quickly with reliability. Therefore any effective analysis is based on First and Second Order Reliability Methods[4] (FORM/SORM).

The basis-variables $\mathbf{X}$ are transformed into standard normal variables $\mathbf{U}$, such that g_u is the corresponding limit state function in $\mathbf{U}$–space. In FORM, a linear approximation V_F of the failure region V is generated, such that the limit state surface ∂V is approximated in a point $\mathbf{u}_0$ by

$$\partial V = \left\{ \mathbf{u} \;\middle|\; \boldsymbol{\alpha}^T \mathbf{u} + \beta = 0 \right\}, \tag{5}$$

with $|\boldsymbol{\alpha}| = 1$. If the limit state function is linear in the $\mathbf{U}$–space, then the failure probability P_f is exactly given by

$$\mathrm{P}_f = \mathrm{P}(\boldsymbol{\alpha}^T \mathbf{U} \le -\beta) = \Phi(-\beta) \tag{6}$$

where Φ is the Gauss distribution function. The failure probability P_f depends only on the safety index β. In limit analysis the limit state function is linear in strength and loading. This favourable situation is preserved in $\mathbf{U}$–space, if both are normally distributed.

The design point $\mathbf{u}^*$ is the solution of the optimization problem

$$\beta = \min\{\mathbf{u}^T\mathbf{u}, \ \ | \ g_u(\mathbf{u}) \le 0\}. \tag{7}$$

Then P_f may be computed from the minimum distance of the limit state surface ∂V from origin in this $\mathbf{U}$–space. The design point is the point $\mathbf{u}^*$ on ∂V, which is the closest to the origin. Failure is most probable for data near the design point. The limit state function $g_u(\mathbf{U})$ is approximated by its linear Taylor series of a point $\mathbf{u}_0 \in \partial V$ in order to generate the iterative procedure

$$\mathbf{u}_{k+1} = \frac{\nabla_u g_u(\mathbf{u}_k)}{|\nabla_u g_u(\mathbf{u}_k)|^2} \left[\mathbf{u}_k^T \nabla_u g_u(\mathbf{u}_k) - g_u(\mathbf{u}_k)\right]. \tag{8}$$

as a search algorithm for $\mathbf{u}^*$. This poses a nonlinear optimization problem which needs the gradient of $g_u(\mathbf{u})$. The derivates are determined by

$$\nabla_u g_u(\mathbf{u}) = \nabla_u g(\mathbf{x}) = \nabla_x g(\mathbf{x}) \nabla_u \mathbf{x}. \tag{9}$$

If the deterministic structural problem is solved in a step-by-step iterative FEM analysis this gradient information is obtained from a sensitivity analysis, which consumes much computing time. Extension of this type of reliability analysis to plastic structural failure faces several already mentioned problems which are not present in linear elastic analysis: local stress has no direct relevance to plastic failure and structural behaviour becomes load-path dependent. No straight-forward $g(\mathbf{X})$ is obtained from standard incremental analysis if failure is assumed by plastic collapse, by ratcheting or by plastic shakedown (LCF). It is even more difficult to obtain the gradient of $g(\mathbf{X})$. Therefore, as an additional draw-back MCS (improved by importance sampling or by some other means of variance reduction) is used in connection with incremental nonlinear reliability analyses.

In FEM, limit and shakedown analysis are formulated as optimization problems. For these optimization problems the Lagrange multipliers of the solution are the strength and the load components of the gradient vector $\nabla_x g(\mathbf{x})$. Thus, no sensitivity analysis is necessary.

3 CONCEPTS OF LIMIT AND SHAKEDOWN ANALYSIS

Static theorems are formulated in terms of stress and define safe structural states by giving an optimization problem for safe loads. The maximum safe load is the limit load avoiding collapse. Alternatively, kinematic theorems are formulated in terms of kinematic quantities and define unsafe structural states yielding a dual optimization problem for the minimum of limit loads. Any admissible solution to the static or kinematic theorem is a true lower or upper bound to the safe load, respectively. Both can be made as close as desired to the exact solution. If upper and lower bound coincide, it could be stated that the true solution has been found.

3.1 Static or lower bound limit load analysis

The limit load factor is defined in (10) by $\alpha = \mathbf{P}_l / \mathbf{P}_0$, where $\mathbf{P}_l = (\mathbf{b}_l, \mathbf{p}_l)$ and $\mathbf{P}_0 = (\mathbf{b}_0, \mathbf{p}_0)$ are the plastic limit load and the chosen reference load, respectively. Here we have supposed that all loads (**b** body forces and **p** surface loads) are applied in a monotone and proportional way. We look for the maximum load factor for which the structure is safe. The structure is safe against plastic collapse if there is a stress field $\boldsymbol{\sigma}$ such that the equilibrium equations are satisfied and the yield condition is nowhere violated. We obtain the following maximum problem:

$$\begin{array}{rl} \max & \alpha \\ \text{s. t.} & \Phi(\boldsymbol{\sigma}) \leq \ \sigma_y \quad \text{in } \Omega \\ & \operatorname{div}\boldsymbol{\sigma} = - \ \alpha \mathbf{b}_0 \quad \text{in } \Omega \\ & \boldsymbol{\sigma} \ \mathbf{n} = \ \alpha \mathbf{p}_0 \quad \text{on } \partial\Omega_\sigma \end{array} \tag{10}$$

for the structure Ω, traction boundary $\partial\Omega_\sigma$ (with outer normal $\mathbf{n}$), yield function Φ, body forces $\alpha \mathbf{b}_0$ and surface loads $\alpha \mathbf{p}_0$.

3.2 Static or lower bound shakedown analysis

The shakedown analysis starts from Melan's lower bound theorem[9]. In the shakedown analysis the equilibrium conditions and the yield criterion for the actual stresses have to be fulfilled at every instant of the load history. We look for the maximum load factor for which the structure is safe. The structure is safe against LCF or ratcheting if there is a stress field $\boldsymbol{\sigma}(t)$ such that the equilibrium equations are satisfied and the yield condition is nowhere and at no instant t violated. We formulate the next maximum problem:

$$\begin{array}{rl} \max & \alpha \\ \text{s. t.} & \Phi(\boldsymbol{\sigma}(t)) \leq \ \sigma_y \quad \text{in } \Omega \\ & \operatorname{div}\boldsymbol{\sigma}(t) = - \ \alpha \mathbf{b}_0(t) \quad \text{in } \Omega \\ & \boldsymbol{\sigma}(t) \ \mathbf{n} = \ \alpha \mathbf{p}_0(t) \quad \text{on } \partial\Omega_\sigma \end{array} \tag{11}$$

for body forces $\alpha \mathbf{b}_0(t)$ and surface loads $\alpha \mathbf{p}_0(t)$, for all $\mathbf{b}_0(t)$, $\mathbf{p}_0(t)$ in a given initial load domain $\mathcal{L}_0$.

In the maximum problems (10) and (11), the actual stresses $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}(t)$ are splitted into fictitious elastic stresses and residual stresses. The deduced problem is solved by a basis reduction technique[5, 13, 16] in the residual stress space and by Sequential Quadratic Programming (SQP).

3.3 Plate with mismatched weld and a centered crack under tension

One half of the thick plate (plane strain) with mismatched weld has the length $L = 40mm$, the width $W = 4mm$, the crack length $a = 2mm$, the thickness B and the weld height $h = 1.2mm$ (see Fig. 2). The different strength data of the base material and the weld material are represented by different yield stresses σ_y^B and σ_y^W, respectively. The main parameter is the mismatch ratio $M = \sigma_y^W/\sigma_y^B$ of yield stress values of base and weld material. A reference value $\sigma_y^B = 100MPa$ of the yield stress is chosen. The example was proposed by the EU-project SINTAP[15] as a benchmark for the Brite-EuRam project LISA: FEM-Based Limit and Shakedown Analysis for Design and Integrity Assessment in European Industry[14] (Project N°: BE 97-4547).

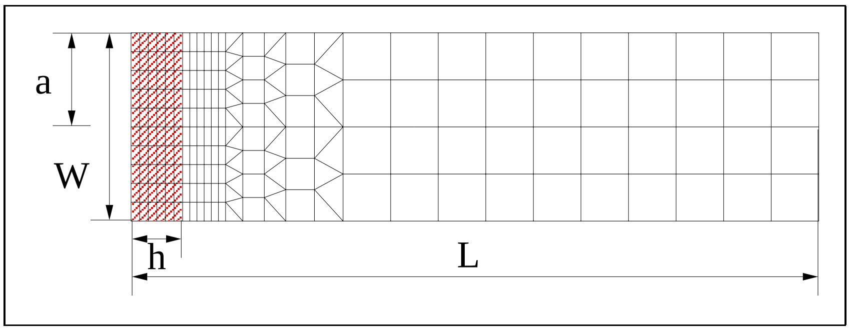


Figure 2: FE mesh

There is a well known exact plane stress limit load F_{yb} for the situation $M = \sigma_y^W/\sigma_y^B = 1$. Estimation of the corresponding plane strain limit load yields the values

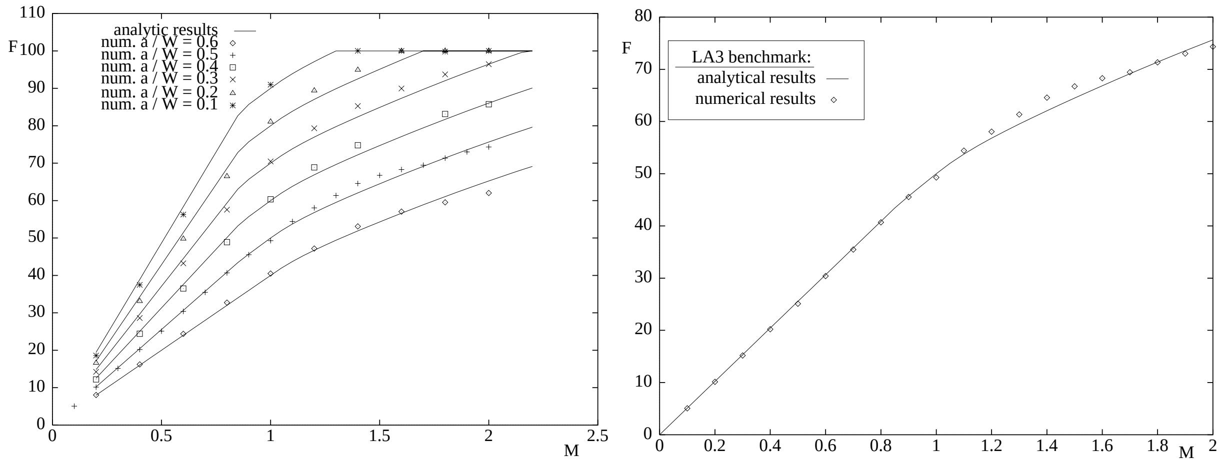
$$\underline{\text{plain stress}}: \; F_{yb} = 2B(W-a)\sigma_y^B \qquad \underline{\text{plain strain}}: \; F_{yb} = \frac{4}{\sqrt{3}}B(W-a)\sigma_y^B \tag{12}$$

Then the applied line load is $50MPa$ and $57.74MPa$, respectively. Approximations for limit load F_{ym} are known[11] for plain stress and strain state. The plane strain results of the direct lower bound FEM approach (using triangular elements) are given in table 1.

Plate with a centered crack in a mismatched weld under tension					
$M = \sigma_y^W/\sigma_y^B$	0.50	0.75	1.00	1.25	1.50
analytic solution[11]	32.33	47.92	57.74	65.82	73.33
lower bound FEM	33.16	49.74	60.38	68.21	75.55

Table 1: Comparison of plane strain limit analysis results

In Fig. 3(a) and (b) analytical and numerical results for plane stress are compared. The limit load is calculated for different values of mismatch factor $M = \sigma_y^W/\sigma_y^B$. For undermatching the



(a) $a/W = 0.1, \ldots, 0.6$, $h/W = 0.3$ (b) $a/W = 0.5$

Figure 3: Comparison of numerical and analytical results of limit load for different crack length and mismatch factor $M = \sigma_y^W / \sigma_y^B$ for plane stress, $h/W = 0.3$

numerical and analytical results in Fig.3(b) are in good agreement (at most 2 % error). In low overmatching case the numerical results differ from the analytical results at most by 4 %.

In the overmatched case the plate fails by the joined plastified regions in the base and the weld material. In the undermatched case the plate fails by plastification of the ligament in the weld, so that the limit load is dominated by the yield stress of the weld material[8, 14] σ_y^W.

4 IMPLEMENTATION AND TEST OF RELIABILITY ANALYSIS

4.1 Implementation

The present contribution uses lower bound theorems of limit and shakedown load to define a limit state function $g(\mathbf{X})$ for reliability analysis by FORM. R and S are respectively defined by the limit or shakedown load factor and the applied load factor. The commercial general purpose FEM code PERMAS[10] is used for discretization. The resulting large-scale optimization problem is transferred to a relatively small one by using a basis reduction method[13, 16].

The solution of the limit load or of shakedown analysis (10) is a linear function of the failure stress σ_y or σ_u if a homogeneous material distribution is assumed. If the structure has a heterogeneous material distribution, we obtain eventually different failure stresses $\sigma_y(x_i)$ or $\sigma_u(x_i)$ in different Gaussian points x_i. Then the limit load is no more a linear function of the failure stresses. In this case the derivates of the limit state function may not be computed directly from the linear function of the failure stresses. The Lagrange multipliers of the optimization problem (10) yield the gradient information of $g(\mathbf{X})$ without any extra computation. This is derived from a distribution theory[5] of $\sigma_u(x_i)$ as the right hand side of (10).

It is most important for the analysis under uncertainty that limit and shakedown analyses are based on a minimum of information concerning the constitutive equations and the load history. This reduces the costs of the collection of statistical data and the need to introduce stochastic models to compensate the lack of data. Due to the so-called tail sensitivity problem there is generally insufficient data to analyze structures of high reliability which are e.g. employed in nuclear reactor technology. Probabilistic limit and shakedown analyses were pioneered in Italy[1]. Further work seemed to remain restricted to stochastic limit analysis of frames based on linear programming[2]. The present contribution extends plastic reliability analysis towards nonlinear programming, shakedown, and a general purpose large-scale FEM approach.

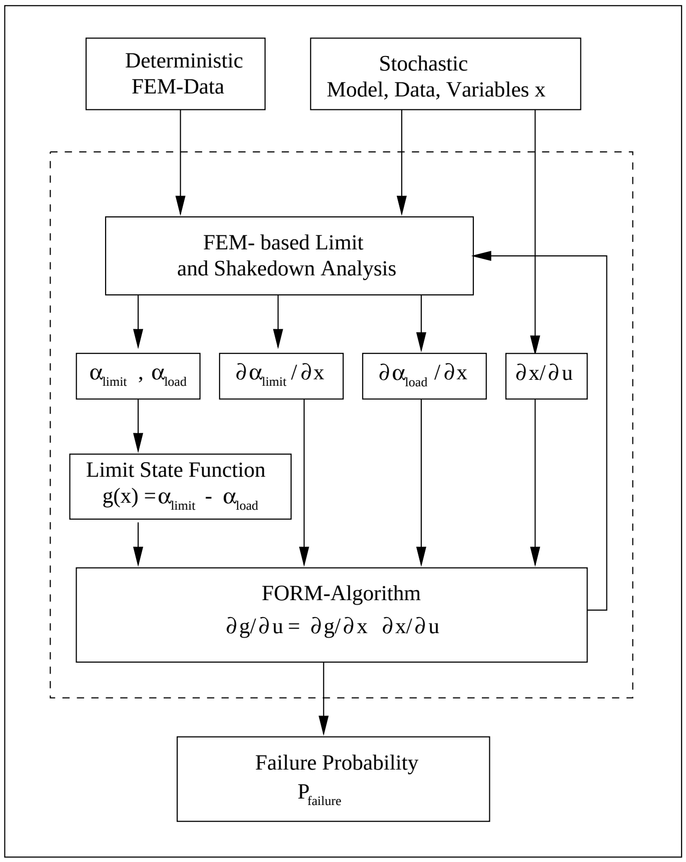


Figure 4: Flowchart of the probabilistic limit load analysis

4.2 Pipe-junction subjected to internal pressure

The pipe-junction[13] under internal pressure p is taken from the collection of PERMAS test examples. It is discretized with 125 solid 27-node hexahedron elements (HEXEC27). The FE-mesh and the essential dimensions of the pipe-junction are represented in Fig. 5. The internal pressure at first yield in the symmetry plane at the inner nozzle corner is calculated to $p_{elastic} \approx 0.0476\sigma_y$. For comparison[13] the limit pressure resulting from the German design rules AD-Merkblatt B9 is calculated to $p_{limit} = 2.85 p_{elastic}$. With the safety factor 1.5 the design pressure is $p_{design} = 1.9 p_{elastic} = 0.0904\sigma_y$.

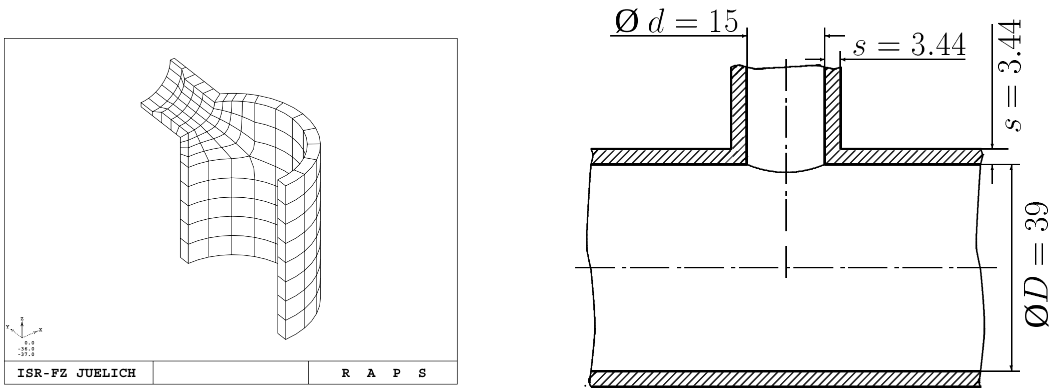


Figure 5: FE-mesh and dimension of a pipe-junction

Numerical limit analysis leads to a collapse pressure of $0.134\sigma_y$. In shakedown analysis the system is subjected to an internal pressure which may vary between zero and a maximum magnitude. The analysis becomes stationary after only 2 iteration steps with the shakedown pressure $p_{SD} = 0.0952\sigma_y$. The shakedown pressure is twice the elastic pressure in good correspondence with an analytic solution[5].

Thus the limit and the shakedown load are linearly dependent of the realization σ_y of the yield stress, which is the basis variable X. The second basis variable Y is the increasing inner pressure P. The limit load P_{lim} of every realization y of $\mathbf{Y}$ is

$$P_{lim}(y) = 0.134y. \tag{13}$$

Obviously, P_{lim} takes the role of a resistance R and P is the loading variable S. The limit state function is defined by

$$g(x, y) = P_{lim} - P = 0.134y - x. \tag{14}$$

The normally distributed random variables X and Y with means μ_x, μ_x and standard deviations σ_x, σ_y, respectively, yield with $x = \sigma_x\, u_x + \mu_x$ and $y = \sigma_y\, u_y + \mu_y$ the transformation

$$g(x, y) = (0.134\mu_y - \mu_x) + 0.134\sigma_y u_y - \sigma_x u_x. \tag{15}$$

With the new random variable $\mathbf{U}$ with realizations $\boldsymbol{u} = (u_x, u_y)^T$, it holds:

$$g(\boldsymbol{u}) = \frac{(-\sigma_x, 0.134\sigma_y)}{\sqrt{\sigma_x^2 + 0.134^2\sigma_y^2}}\, \boldsymbol{u} + \frac{0.134\mu_y - \mu_x}{\sqrt{\sigma_x^2 + 0.134^2\sigma_y^2}}, \tag{16}$$

such that the safety index β of the random variable $\mathbf{U}$ is

$$\beta = \frac{0.134\mu_y - \mu_x}{\sqrt{\sigma_x^2 + 0.134^2\sigma_y^2}} = \frac{0.134\mu_y - \mu_x}{\sqrt{\sigma_x^2 + 0.018\sigma_y^2}} \tag{17}$$

In Figure 6 the numerical results of the shakedown analysis are compared with the analytic values resulting from the exact solution. The results are normalized to the mean values μ_x and μ_y of the corresponding distributions. Both variables are normally distributed with standard deviations $\sigma_x = 0.1\mu_x$ and $\sigma_y = 0.1\mu_y$.

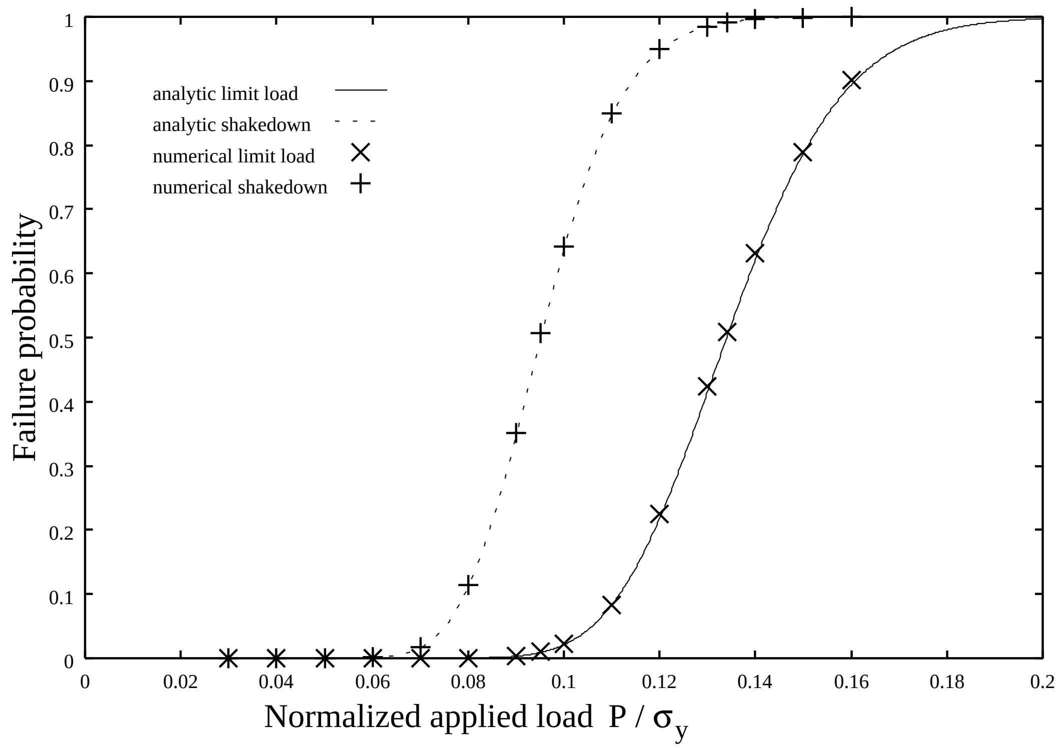


Figure 6: Comparison of numerical with analytical results for $\sigma_x = 0.1\mu_x$, $\sigma_y = 0.1\mu_y$

The results correspond well with the analytic results and demonstrate that reliability analysis can be performed for realistic model sizes at very low computing times compared to incremental analyses. Note, that the latter cannot be used in a quantitative comparison because incremental nonlinear analysis fails to give a sharp evidence for plastic failure.

	Limit analysis		Shakedown analysis	
P/σ_y	P_f (numer.)	P_f (anal.)	P_f (numer.)	P_f (anal.)
0.03	1.8653E-14	1.8135E-14	3.4294E-11	3.2430E-11
0.04	1.1458E-11	8.9725E-12	4.7844E-08	4.5052E-08
0.05	2.2948E-09	2.1383E-09	1.3919E-05	1.3145E-05
0.06	2.5188E-07	2.3252E-07	9.2428E-04	8.7985E-04
0.07	1.2282E-05	1.1513E-05	1.7126E-02	1.6478E-02
0.08	2.7486E-04	2.6997E-04	1.1388E-01	1.1078E-01
0.09	3.3817E-03	3.2069E-03	3.5179E-01	3.4571E-01
0.0952	9.6429E-03	9.1261E-03	5.0654E-01	5.0000E-01
0.1	2.2190E-02	2.1001E-02	6.4212E-01	6.3594E-01
0.11	8.3328E-02	8.3125E-02	8.4933E-01	8.4550E-01
0.12	2.2510E-01	2.1819E-01	9.4897E-01	9.4728E-01
0.13	4.2411E-01	4.1517E-01	9.8519E-01	9.8460E-01
0.134	5.0892E-01	5.0000E-01	9.9113E-01	9.9087E-01
0.14	6.3079E-01	6.2157E-01	9.9610E-01	9.9592E-01
0.15	7.8917E-01	7.8683E-01	9.9902E-01	9.9898E-01
0.16	9.0053E-01	8.9358E-01	9.9976E-01	9.9974E-01

Table 2: Comparison of numerical and analytical results for $\sigma_x = 0.1\mu_x$, $\sigma_y = 0.1\mu_y$

5 CONCLUSIONS

Limit and shakedown theorems of plastic structural failure provide unique definitions of limit state functions. In combination with FEM and with FORM, failure probabilities of passive components are obtained with sufficient precision from a minimum of stochastic data at low computational efforts. Sensitivities need no extra FEM analysis. The remaining numerical error may be estimated or reduced by the additional use of upper bound theorems. Further research is also addressed to more realistic material modeling including two-surface plasticity and continuum damage.

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