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FINITE ELEMENT SHAKEDOWN AND LIMIT RELIABILITY ANALYSIS OF THIN SHELLS

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Summary. A procedure for the evaluation of the failure probability of elastic-plastic thin shell structures is presented. The procedure involves a deterministic limit and shakedown analysis for each probabilistic iteration which is based on the kinematical approach and the use the exact Ilyushin yield surface. Based on a direct definition of the limit state function, the non-linear problems may be efficiently solved by using the First and Second Order Reliability Methods (FORM/SORM). This direct approach reduces considerably the necessary knowledge of uncertain technological input data, computing costs and the numerical error.

1 INTRODUCTION

In the design-by-analysis context, the need to account for uncertainties has long been recognized in order to achieve reliable design of structural and mechanical systems. There is a general agreement that advanced computational tools have to be employed to provide the necessary computational framework for describing structural response and reliability. Current structural reliability analysis is typically based on the limit state of initial or local failure. However, this gives quite pessimistic reliability estimates, because virtually all structures are redundant or statically undetermined. Progressive member failures of such systems reduce redundancy until finally the statically determined system fails. The more effective method of structural reliability analysis is probabilistic limit and shakedown analyses, which is based on the direct computation of the load-carrying capacity or the safety margin.

This paper presents an algorithm of probabilistic limit and shakedown analysis for thin plates and shells, which is based on the kinematical approach. The loading and material strength are to be considered as random variables. Non-linear sensitivity analyses are performed with FORM/SORM in order to get the failure probability of the structure.

2 DETERMINISTIC LIMIT AND SHAKEDOWN ANALYSIS

Let h be the shell thickness and σ_y the uniaxial yield stress. For our purpose to deal with the probabilistic problem, the yield limit σ_y and the loads are considered as random variables. Let us restrict ourselves to the case of homogeneous material such that $\sigma_y = Y\sigma_0$ where σ_0 is a constant reference value and Y is a random variable. The non-dimensional 'engineering' strain resultant vector is introduced as follows

$$\tilde{\boldsymbol{\varepsilon}}=\begin{bmatrix}\bar{\mathbf{e}} & \mathbf{k}\end{bmatrix}^T,\ \bar{\mathbf{e}}=\frac{1}{\varepsilon_0}\begin{bmatrix}\bar{\varepsilon}_{11} & \bar{\varepsilon}_{22} & 2\bar{\varepsilon}_{12}\end{bmatrix}^T,\ \mathbf{k}=\frac{1}{\kappa_0}\begin{bmatrix}\kappa_{11} & \kappa_{22} & 2\kappa_{12}\end{bmatrix}^T \tag{1}$$

in which $N_0 = \sigma_0 h$, $M_0 = \sigma_0 h^2/4$, $\varepsilon_0 = \sigma_0(1-\nu^2)/E$ and $\kappa_0 = 4\varepsilon_0/h$ are the normalized quantities. The quadratic strain intensities are defined in terms of non-dimensional stress resultants by

$$\begin{aligned}
&P_\varepsilon = \frac{3}{4}\left(d\bar{\mathbf{e}}^p\right)^T \mathbf{P}^{-1} d\bar{\mathbf{e}}^p = \frac{3}{4}\left(d\tilde{\boldsymbol{\varepsilon}}^p\right)^T \mathbf{P}_1 d\tilde{\boldsymbol{\varepsilon}}^p \qquad (\geq 0)\\
&P_{\varepsilon\kappa} = 3\left(d\bar{\mathbf{e}}^p\right)^T \mathbf{P}^{-1} d\mathbf{k}^p = 3\left(d\tilde{\boldsymbol{\varepsilon}}^p\right)^T \mathbf{P}_2 d\tilde{\boldsymbol{\varepsilon}}^p\\
&P_\kappa = 12\left(d\mathbf{k}^p\right)^T \mathbf{P}^{-1} d\mathbf{k}^p = 12\left(d\tilde{\boldsymbol{\varepsilon}}^p\right)^T \mathbf{P}_3 d\tilde{\boldsymbol{\varepsilon}}^p \qquad (\geq 0)
\end{aligned} \tag{2}$$

where $\mathbf{P}^{-1}$, $\mathbf{P}_i$ $(i = 1,2,3)$ are

$$\mathbf{P}^{-1} = \begin{pmatrix} 4/3 & 2/3 & 0\\ 2/3 & 4/3 & 0\\ 0 & 0 & 1/3\end{pmatrix},\ \mathbf{P}_1 = \begin{pmatrix}\mathbf{P}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix},\ \mathbf{P}_2 = \begin{pmatrix}\mathbf{0} & \mathbf{P}^{-1}/2\\ \mathbf{P}^{-1}/2 & \mathbf{0}\end{pmatrix},\ \mathbf{P}_3 = \begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{P}^{-1}\end{pmatrix} \tag{3}$$

In order to avoid the difficulties arising with the parameterization of the Ilyushin yield surface and to use the exact yield surface in practical computations, Burgoyne and Brennan[1] introduced the parameters

$$\upsilon = \frac{P_\varepsilon}{P_\kappa},\ \beta = -\frac{P_{\varepsilon\kappa}}{P_\kappa},\ \gamma = \upsilon - \beta^2,\ \beta_1 = 0.5 - \beta,\ \beta_2 = 0.5 + \beta \tag{4}$$

With these parameters, the plastic dissipation function for a shell structure may be written in the form[2]

$$D^p(\dot{\tilde{\boldsymbol{\varepsilon}}}^p) = YN_0\varepsilon_0\sqrt{\frac{P_\kappa}{3}}\left(\beta_1\sqrt{\beta_1^2+\gamma} + \beta_2\sqrt{\beta_2^2+\gamma} + \gamma K_0\right) \tag{5}$$

Consider a convex polyhedral load domain $\mathcal{L}$ and a special loading path consisting of all load vertices $\hat{P}_k$ $(k = 1,\dots,m)$ of $\mathcal{L}$. By discretizing the whole structure by finite elements and application of Koiter's theorem, the shakedown limit, which is the smaller one of the low cycle fatigue limit, and the ratcheting limit may be found by the following minimization

$$\alpha_{\lim} = \min \sum_{k=1}^{m} \sum_{i=1}^{NG} w_i Y N_0 \varepsilon_0 \sqrt{\frac{P_\kappa}{3}} \left(\beta_1 \sqrt{\beta_1^2 + \gamma} + \beta_2 \sqrt{\beta_2^2 + \gamma} + \gamma K_0 \right)$$

$$\text{s.t.:} \quad \begin{cases} \sum_{k=1}^{m} \dot{\tilde{\boldsymbol{\varepsilon}}}_{ik} = \mathbf{B}_i \dot{\mathbf{u}} & \forall i = 1,\ldots,NG \\ \sum_{k=1}^{m} \sum_{i=1}^{NG} w_i N_0 \varepsilon_0 \dot{\tilde{\boldsymbol{\varepsilon}}}_{ik}^T \tilde{\boldsymbol{\sigma}}_{ik}^E = 1 \end{cases} \tag{6}$$

By applying Newton's method to solve the KKT conditions of system (6) we obtain the Newton directions $d\dot{\mathbf{u}}$ and $d\dot{\mathbf{e}}_{ik}$, which assure that a suitable step along them will lead to a decrease of the objective function α^+. If the relative improvement between two steps is smaller than a given constant, the algorithm stops and leads to the shakedown limit factor.

3 PROBABILISTIC LIMIT AND SHAKEDOWN ANALYSIS

In this paper FORM/SORM are used for structural reliability analysis. The limit state function separating the safe and failure regions is defined directly as the difference between the obtained limit load factor and the current load factor. If we defined the limit load factor $\alpha_{\lim} = P_{\lim} / P$ where $P_{\lim}$, P are limit load and actual load of the structure, then after being normalized with the actual load P, the limit state function becomes $g = \alpha_{\lim} - 1$. The calculation of the design point in the standard Gaussian space $\mathbf{u}$ leads to a non-linear constrained optimization problem as follows[3]

$$\begin{aligned} &\text{minimize:} \quad f(\mathbf{u}) = \frac{1}{2}\mathbf{u}^T\mathbf{u} \\ &\text{s.t.} \qquad\quad g(\mathbf{u}) \le 0 \end{aligned} \tag{8}$$

A general method called the Sequential Quadratic Programming (SQP) was used to solve the above problem. This method has proved to be suitable for tasks in the area of the reliability theory After each deterministic step, the Jacobian and the Hessian of the limit state function, which are needed for FORM/SORM, are obtained directly from a mathematical analysis with no extra computational cost[4].

4 NUMERICAL APPLICATION

Consider a square plate with central circular hole which has the ratio $R/L = 0.2$ and is subjected to a uniaxial tension p which can be constant or can vary within the range $[0, p_{\max}]$. Both yield limit σ_y and load p are supposed to be log-normal distribution random variables. For this case, the exact plastic collapse limit and the numerical shakedown load are given by $p_{\lim} = (1 - R/L)\sigma_y$, $p_{sh} = 0.60332\sigma_y$. Numerical results of failure probabilities P_f are introduced in tables 1, compared with the exact solutions[3]. The FORM and SORM results are identical within 4 digits which demonstrates the linearity of the limit state function in the shakedown approach.

Limit analysis			Shakedown analysis		
μ_p / μ_{σ_y}	P_f (anal.)	P_f (FORM/SORM)	μ_p / μ_{σ_y}	P_f (anal.)	P_f (FORMSORM)
			0.2	2.501E-15	1.817E-15
0.3	1.790E-12	1.704E-12	0.3	3.661E-07	3.404E-07
0.4	4.473E-07	5.097E-07	0.4	1.788E-03	1.533E-03
0.5	4.315E-04	5.205E-04	0.5	9.151E-02	8.708E-02
0.6	2.071E-02	1.814E-02	0.6	4.844E-01	4.716E-01
			0.60332	5.000E-01	4.883E-01
0.7	1.719E-01	1.604E-01	0.7	8.540E-01	8.481E-01
0.8	5.000E-01	4.794E-01	0.8	9.773E-01	9.760E-01
0.9	7.981E-01	7.839E-01	0.9	9.977E-01	9.976E-01
1.0	9.431E-01	9.373E-01	1.0	9.998E-01	9.998E-01
1.1	9.880E-01	9.867E-01	1.1	9.999E-01	9.999E-01
1.2	9.979E-01	9.976E-01			
1.3	9.997E-01	9.997E-01			

Table 1: Numerical results and comparison for log-normal distributions with $\sigma_{p,\sigma} = 0.1\mu_{p,\sigma}$

5 CONCLUSIONS

The advantages of this method are that the failure under cyclic loading is treated as a time-invariant problem and that the limit state function becomes (nearly) linear. Moreover, sensitivity analyses are obtained directly from the mathematical optimization solution of the deterministic problem with no extra computational cost. The proposed method appears to be capable of identifying good estimates of the failure probability, even in the case of very small probabilities.

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