**7th International Conference
on Reliability of Materials and Structures (RELMAS 2008)
June 17 – 20, 2008
Saint Petersburg, Russia**

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LIMIT AND SHAKEDOWN RELIABILITY ANALYSIS BY NONLINEAR PROGRAMMING

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INTRODUCTION

Analysis of advanced structures working under extreme heavy loading such as nuclear power plants and piping system should take into account the randomness of loading, geometrical and material parameters. The existing reliability are restricted mostly to the elastic working regime, e.g. allowable local stresses. Development of the limit and shakedown reliability-based analysis and design methods, exploiting potential of the shakedown working regime, is highly needed.

In this paper the application of a new algorithm of probabilistic limit and shakedown analysis for shell structures is presented, in which the loading and strength of the material as well as the thickness of the shell are considered as random variables. The reliability analysis problems may be efficiently solved by using a system combining the available FE codes, a deterministic limit and shakedown analysis, and the First and Second Order Reliability Methods (FORM/SORM). Non-linear sensitivity analyses are obtained directly from the solution of the deterministic problem without extra computational costs.

DETERMINISTIC LIMIT AND SHAKEDOWN ANALYSIS

Let us restrict ourselves to the case of homogeneous material and shells of constant thickness in which the yield limit σ_y and thickness h are the same at every Gaussian point of the structure. So we always can write $\sigma_y = Y\sigma_0$, $h = Zh_0$ where σ_0, h_0 are constant reference values and Y, Z are random variables. The dimensionless 'engineering' stress and strain resultant vector are introduced as follows

$$\begin{aligned} &\tilde{\boldsymbol{\sigma}}=\begin{bmatrix}\mathbf{n} & \mathbf{m}\end{bmatrix}^T, \quad \mathbf{n}=\frac{1}{N_0}\begin{bmatrix}N_{11} & N_{22} & N_{12}\end{bmatrix}^T, \quad \mathbf{m}=\frac{1}{M_0}\begin{bmatrix}M_{11} & M_{22} & M_{12}\end{bmatrix}^T, \\ &\tilde{\boldsymbol{\varepsilon}}=\begin{bmatrix}\bar{\mathbf{e}} & \mathbf{k}\end{bmatrix}^T, \quad \bar{\mathbf{e}}=\frac{1}{\varepsilon_0}\begin{bmatrix}\bar{\varepsilon}_{11} & \bar{\varepsilon}_{22} & 2\bar{\varepsilon}_{12}\end{bmatrix}^T, \quad \mathbf{k}=\frac{1}{\kappa_0}\begin{bmatrix}\kappa_{11} & \kappa_{22} & 2\kappa_{12}\end{bmatrix}^T, \end{aligned} \tag{1}$$

where $N_0 = \sigma_0 h_0$, $M_0 = \sigma_0 h_0^2/4$, $\varepsilon_0 = \sigma_0(1-\nu^2)/E$ and $\kappa_0 = 4\varepsilon_0/h_0$ are the normalized quantities. The quadratic strain intensities are defined in terms of the incremental 'engineering' strain resultants by

$$\begin{aligned} P_\varepsilon &= \frac{3}{4}\left(d\bar{\mathbf{e}}^p\right)^T \mathbf{P}^{-1} d\bar{\mathbf{e}}^p = \frac{3}{4}\left(d\tilde{\boldsymbol{\varepsilon}}^p\right)^T \mathbf{P}_1 d\tilde{\boldsymbol{\varepsilon}}^p \qquad (\geq 0) \\ P_{\varepsilon\kappa} &= 3\left(d\bar{\mathbf{e}}^p\right)^T \mathbf{P}^{-1} d\mathbf{k}^p = 3\left(d\tilde{\boldsymbol{\varepsilon}}^p\right)^T \mathbf{P}_2 d\tilde{\boldsymbol{\varepsilon}}^p \\ P_\kappa &= 12\left(d\mathbf{k}^p\right)^T \mathbf{P}^{-1} d\mathbf{k}^p = 12\left(d\tilde{\boldsymbol{\varepsilon}}^p\right)^T \mathbf{P}_3 d\tilde{\boldsymbol{\varepsilon}}^p \qquad (\geq 0) \end{aligned} \tag{2}$$

where $\mathbf{P}^{-1}$, $\mathbf{P}_i$ $(i = 1,2,3)$ are

$$\mathbf{P}^{-1} = \begin{pmatrix} 4/3 & 2/3 & 0 \\ 2/3 & 4/3 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}, \quad \mathbf{P}_1 = \begin{pmatrix} \mathbf{P}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad \mathbf{P}_2 = \begin{pmatrix} \mathbf{0} & \mathbf{P}^{-1}/2 \\ \mathbf{P}^{-1}/2 & \mathbf{0} \end{pmatrix}, \quad \mathbf{P}_3 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}^{-1} \end{pmatrix}. \tag{3}$$

In order to avoid the difficulties arising with the parameterization of the Ilyushin yield surface [1] and to use the exact yield surface in practical computations, Burgoyne and Brennan [2] introduced the parameters

$$\upsilon = \frac{P_\varepsilon}{P_\kappa}, \quad \beta = -\frac{P_{\varepsilon\kappa}}{P_\kappa}, \quad \gamma = \upsilon - \beta^2, \quad \beta_1 = 0.5 - \beta, \quad \beta_2 = 0.5 + \beta, \quad K_0 = \ln\left(\frac{\sqrt{\beta_1^2 + \gamma} + \beta_1}{\sqrt{\beta_2^2 + \gamma} - \beta_2}\right). \tag{4}$$

With these parameters, the plastic dissipation function for a shell structure may be written in the form [3]

$$D^p(\dot{\boldsymbol{\varepsilon}}^p) = YN_0\varepsilon_0\sqrt{\frac{P_\kappa}{3}}\left(\beta_1\sqrt{\beta_1^2+\gamma}+\beta_2\sqrt{\beta_2^2+\gamma}+\gamma K_0\right). \tag{5}$$

Consider a convex polyhedral load domain $\mathcal{L}$ and a special loading path consisting of all load vertices $\hat{P}_k$ ($k=1,...,m$) of $\mathcal{L}$. By discretizing the whole structure by finite elements and application of Koiter's upper bound theorem, the shakedown limit, which is the smaller one of the low cycle fatigue limit, and the ratchetting limit may be found by the following nonlinear minimization

$$\begin{aligned}
&\alpha_{\lim} = \min\sum_{k=1}^{m}\sum_{i=1}^{NG} w_i YN_0\varepsilon_0\sqrt{\frac{P_\kappa}{3}}\left(\beta_1\sqrt{\beta_1^2+\gamma}+\beta_2\sqrt{\beta_2^2+\gamma}+\gamma K_0\right)\\
&\text{s.t.:}\quad \begin{cases}\displaystyle\sum_{k=1}^{m}\dot{\tilde{\boldsymbol{\varepsilon}}}_{ik} = \mathbf{B}_i\dot{\mathbf{u}} & \forall i=1,...,NG\\ \displaystyle\sum_{k=1}^{m}\sum_{i=1}^{NG} w_i N_0\varepsilon_0\dot{\tilde{\boldsymbol{\varepsilon}}}_{ik}^T\tilde{\boldsymbol{\sigma}}_{ik}^E = 1 & \end{cases}
\end{aligned} \tag{6}$$

By applying Newton's method to solve the KKT (Karush-Kuhn-Tucker) conditions of system (6) we obtain the Newton directions $d\dot{\mathbf{u}}$ and $d\dot{\tilde{\boldsymbol{\varepsilon}}}_{ik}$, which assure that a suitable step along them will lead to a decrease of the objective function. If the relative improvement between two steps is smaller than a given constant, the algorithm stops and leads to the shakedown limit factor $\alpha_{\lim}$. Details of the iterative algorithm can be found in [3].

LIMIT AND SHAKEDOWN RELIABILITY-BASED ANALYSIS

In structural reliability analysis, the limit state function which is based on the comparison of a structural resistance (threshold) and loading, defines the limit state hyper-surface which separates the failure region from the safe region. Thus, the failure probability is the probability that the limit state function is non-positive. In general, it is impossible to calculate the failure probability analytically and therefore, approximation approaches should be used. By FORM an approximation to the probability of failure is obtained by linearising the limit state function at the 'design point'. This is the point on the limit state surface that is nearest to the origin in the space of standard normal random variables. Due to the exponential decay of the probability density, the design point has the highest probability among all points in the failure domain. It follows that the neighborhood of this point makes the dominant contribution to the failure probability. SORM improves on the FORM approximation by using a quadratic hypersurface fitted at the design point to the limit state surface.

If we defined the shakedown load factor which is obtained from the nonlinear optimization problem in Eq. (6) as $\alpha_{\lim} = P_{\lim}/P$ where $P_{\lim}$ and P are limit load and actual load of the structure respectively, then after being normalized with the actual load P, the limit state function becomes $g = \alpha_{\lim} - 1$. The calculation of design point in the standard Gaussian space $\mathbf{u}$ leads to a non-linear constrained optimization problem as follows [4]

$$\begin{aligned} &\text{minimize:} \quad f(\mathbf{u}) = \frac{1}{2}\mathbf{u}^T\mathbf{u} \\ &\text{s.t.} \qquad\quad g(\mathbf{u}) \le 0 \end{aligned} \tag{7}$$

A general method called the Sequential Quadratic Programming (SQP) was used to solve the above problem. This method has proved to be suitable for tasks in the area of the reliability theory. After each deterministic step, the Jacobian and the Hessian of the limit state function, which are needed for FORM/SORM, are obtained directly from the mathematical analysis of the deterministic shakedown problem with no extra computational cost [5].

Numerical application

A pipe bend is investigated with the random variables in-plane bending moment M_I, and wall thickness h, with mean values μ_s, μ_t and standard deviations σ_s, σ_t respectively. The failure probabilities P_f are calculated for different distributions of random variables, and they are presented in the figure 1 and figure 2 versus μ_s/μ_t.

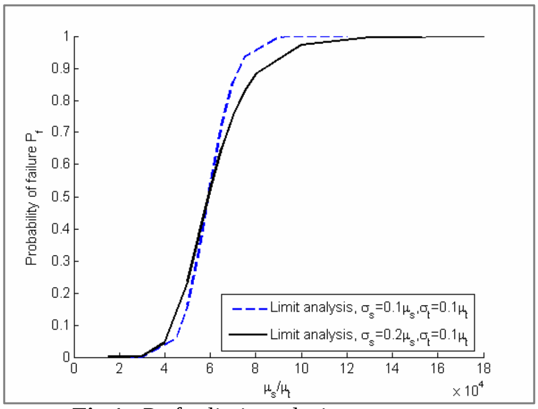

Fig 1. P_f for limit analysis, normally distributed variables.

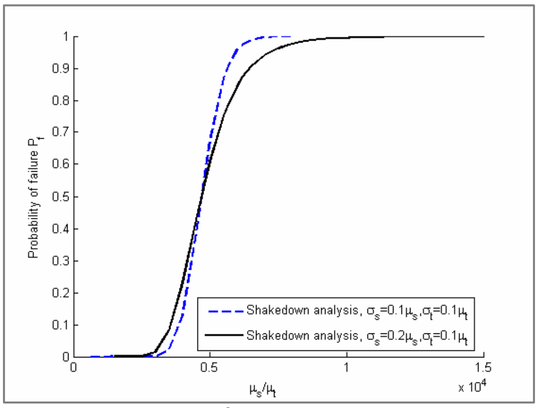

Fig 2. P_f for shakedown analysis, normally distributed variables.

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