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A MULTICRITERIA METHOD FOR TRUSS OPTIMIZATION

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Abstract. We propose a multicriteria approach to the optimization of trusses. Stress and cross sectional areas of bars are considered as independent variables. This is a distinguishing point of this method. Using two independent variables makes the equilibrium equations become nonlinear. The problem is solved by the Optimization Toolbox of Matlab. In this paper, we consider the nine-bar truss of as an example.

1 INTRODUCTION

The most popular method to treat problems of elastic analysis, limit analysis, shakedown analysis and problem of optimal design is with an optimization loop over the solution of the structural problem. The structural problem can be formulated as a convex optimization problem. In this work, we present an approach which combines both optimizations in a multicriteria problem. We demonstrate the idea for the example of truss structure using stresses as variables. The condition of the minimum of the strain energy is adopted to give enough equations for solving the problem.

2 ELASTIC ANALYSIS OF TRUSSES

We consider linear truss (no change of the angle between bars after deformation), if stresses or internal forces of the bar are considered as unknowns, then the elastic analysis problem is written as:

min (*the strain energy in terms of the member forces or the stress form*) (1)

s.t.: *the equilibrium equations written for nodes having Degree of Freedom*

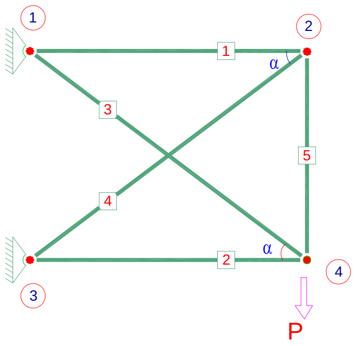

Figure1: A truss structure

To understand the above formulation, without loss of generality, we consider this following simple example: a truss is subjected to an external load P as shown in Figure 1. We wish to find the member forces N_1, N_2, N_3, N_4, N_5 of the trusses. The problem for the elastic analysis of trusses is now considered as an optimization problem as follows:

$$\min \frac{1}{2EA}(N_1^2 l_1 + N_2^2 l_2 + N_3^2 l_3 + N_4^2 l_4 + N_5^2 l_5) \tag{2}$$

$$\text{s.t.}: \begin{cases} N_1 + N_4 \cos\alpha = 0, & N_5 + N_4 \sin\alpha = 0, \\ N_2 + N_3 \cos\alpha = 0, & N_5 + N_3 \sin\alpha = P. \end{cases}$$

By using the Lagrange multiplier method, we can write:

$$\begin{aligned} L(N,\lambda) = & \frac{1}{2EA}\left(N_1^2 l_1 + N_2^2 l_2 + N_3^2 l_2 + N_4^2 l_4 + N_5^2 l_5\right) \\ & + \lambda_1\left(N_1 + N_4 \cos\alpha\right) + \lambda_2\left(N_5 + N_4 \sin\alpha\right) + \lambda_3(N_2 + N_3 \cos\alpha) + \lambda_4(N_5 + N_3 \sin\alpha - P) \end{aligned} \tag{3}$$

where λ_i are the Lagrange multipliers and are unknowns as well. The condition of the minimum of (3) is written as:

$$\begin{aligned} & \frac{\partial L}{\partial N_1} = \frac{N_1 l_1}{EA} + \lambda_1 = 0\,; \quad \frac{\partial L}{\partial N_2} = \frac{N_2 l_2}{EA} + \lambda_3 = 0\,; \quad \frac{\partial L}{\partial N_3} = \frac{N_3 l_3}{EA} + \lambda_4 \sin\alpha = 0 \\ & \frac{\partial L}{\partial N_4} = \frac{N_4 l_4}{EA} + \lambda_1 \cos\alpha + \lambda_2 \sin\alpha = 0\,; \quad \frac{\partial L}{\partial N_5} = \frac{N_5 l_5}{EA} + \lambda_2 + \lambda_4 = 0 \\ & \frac{\partial L}{\partial \lambda_1} = N_1 + N_4 \cos\alpha = 0\,; \quad \frac{\partial L}{\partial \lambda_2} = N_5 + N_4 \sin\alpha = 0 \\ & \frac{\partial L}{\partial \lambda_3} = N_2 + N_3 \cos\alpha = 0\,; \quad \frac{\partial L}{\partial \lambda_4} = N_5 + N_3 \sin\alpha = P \end{aligned} \tag{4}$$

The whole system has 9 equations with 9 unknowns, which allow us to determine member forces N_1, N_2, N_3, N_4, N_5 and λ_i. It can be easily seen that λ_1, λ_2 are the displacements of node 2 and λ_3, λ_4 are the displacements of node 4 in x and y directions, respectively. The elastic problem is now analyzed by solving the above nine equations to get internal force of bars and the displacements of nodes as well. Solving system of the above nine equations is equivalent to deal with problem (1). Afterwards, we will treat directly the problem of elastic analysis by analyzing problem (1).

3 MINIMUM VOLUME OF TRUSSES AT ELASTIC STATE

Consider linear trusses having n bars. l_i, t_i, s_i are the length, cross sectional area and stress in i^{th} bar. In this problem, the objective function is the minimum of volume. Constrains include the minimum of the strain energy, the equilibrium equations written for the nodes with degrees of freedom, stresses of bars not exceeding an allowable stress and the condition for non-negative of cross sectional areas. The formulation is as follows:

$$\min \sum_{i=1}^{n} l_i t_i \tag{5}$$

$$\text{s.t.:} \quad \begin{cases} \min \sum_{i=1}^{n} \frac{1}{2E} s_i^2 t_i l_i & \text{(a)} \\ \left[\sum_{i=1}^{n_{ij}} s_i t_i \cos\alpha_{ix} - \mu f_{jx} = 0 \right]_{jx} \qquad jx = 1,2,\ldots,n_{jx} & (b) \\ \left[\sum_{i=1}^{n_{ij}} s_i t_i \sin\alpha_{ix} - \mu f_{jy} = 0 \right]_{jy} \qquad jy = 1,2,\ldots,n_{jy} & (c) \\ -s_a \le s_i \le s_a & (d) \\ t_i \ge 0 & (e) \end{cases}$$

where α_{ix} is the angle between $i-th$ bar with x-axis, $\alpha_{ix} \le \pi/2$;

f_{jx}, f_{jy} are applied loads acting on $j-th$ node in x and y direction;

n_{jx}, n_{jy} are number of nodes having degrees of freedom on x and y directions, respectively

n_{ij} is number of bars connected to *j-th* node

E is the Young's modulus, s_a is the allowable stress of the material

As the stresses and cross sectional areas of each bar are independent parameters, the truss optimization problem at elastic state is written as follows:

$$\min \sum_{i=1}^{n} l_i t_i + \min \sum_{i=1}^{n} \frac{1}{2E} s_i^{\,2} t_i l_i \tag{6}$$

$$\text{s.t.:} \begin{cases} \left[\sum_{i=1}^{n_{ij}} s_i t_i \cos \alpha_{ix} - \mu f_{jx} = 0 \right]_{jx} & jx = 1,2,\ldots,n_{jx} & (a) \\ \left[\sum_{i=1}^{n_{ij}} s_i t_i \sin \alpha_{ix} - \mu f_{jy} = 0 \right]_{jy} & jy = 1,2,\ldots,n_{jy} & (b) \\ -s_a \le s_i \le s_a & & (c) \\ t_i \ge 0 & & (d) \end{cases}$$

The equilibrium equations (a,b) are nonlinear. It is emphasized that the above problem is a multicriteria problem [p181, 5].

4 PLASTIC LIMIT ANALYSIS

In the problem of limit analysis, loads applied on structures increase proportionally. Limit loads are unknown. The yield limits of trusses are given by the tensile yield stress and compressive yield stress, for simplicity, we assume that they have the same magnitude, s_0. According to lower bound theorem, similar to the problem (5), this problem is formulated as:

$$\max \mu \left[\sum_{jx=1}^{n_{jx}} f_{jx} + \sum_{jx=1}^{n_{jy}} f_{jy} \right] + \min \sum_{i=1}^{n} \frac{1}{2E} s_i^2 t_0 l_i \tag{7}$$

$$\text{s.t.:} \begin{cases} \left[\sum_{i=1}^{n_{ij}} s_i t_0 \cos \alpha_{ix} - \mu f_{jx} = 0 \right]_{jx} & jx = 1,2,\ldots,n_{jx} & (a) \\ \left[\sum_{i=1}^{n_{ij}} s_i t_0 \sin \alpha_{ix} - \mu f_{jy} = 0 \right]_{jy} & jy = 1,2,\ldots,n_{jy} & (b) \\ -s_0 \le s_i \le s_0 & & (c) \end{cases}$$

Where μ is the load factor; s_0 is the yield stress of the material and t_0 is the given cross sectional area which is assumed to be the same for all bars.

5 MINIMUM VOLUME OF TRUSSES AT PLASTIC LIMIT STATE

In this problem, the cross sectional area of bar and limit load are unknowns. Therefore, instead of using the condition $\min V$, we use another weak condition in the constraints: the volume of material of the truss is less than a given source of material V_0. In the formulation of the lower bound theorem, we can write:

$$\max \mu\left[\sum_{jx=1}^{n_{jx}} f_{jx} + \sum_{jx=1}^{n_{jy}} f_{jy}\right] + \min \sum_{i=1}^{n} \frac{1}{2E} s_i^2 t_i l_i \tag{8}$$

$$\text{s.t.:}\quad \begin{cases} \left[\sum_{i=1}^{n_{ij}} s_i t_i \cos\alpha_{ix} - \mu f_{jx} = 0\right]_{jx} & jx = 1,2,\dots,n_{jx} & (a) \\ \left[\sum_{i=1}^{n_{ij}} s_i t_i \sin\alpha_{ix} - \mu f_{jy} = 0\right]_{jy} & jy = 1,2,\dots,n_{jy} & (b) \\ -s_0 \le s_i \le s_0 & & (c) \\ 0 \le t_i & & (d) \\ \sum_{i=1}^{n} t_i l_i \le V_0 & & (e) \end{cases}$$

where μ is the load factor. It can be seen in the later examples, if V_0 increases then the limit load increases.

6 SHAKEDOWN ANALYSIS.

If a structure has been designed for a given load variation domain with plastic range of material response taken into account and plastic deformation range is stable, we have a shakedown problem. In case of a truss subjected to a load domain with *NV* vertices, according to the lower bound theorem the problem is formulated as:

$$\max \sum_{L=1}^{NV} \mu\left[\sum_{jx=1}^{n_{jx}} f_{jxL} + \sum_{jy=1}^{n_{jy}} f_{jyL}\right] + \min \sum_{L=1}^{NV}\left[\sum_{i=1}^{n} \frac{1}{2E} s_i^2 t_0 l_i\right] \tag{9}$$

$$\text{s.t.:}\quad \begin{cases} \sum_{L=1}^{NV}\left[\sum_{i=1}^{n_{ij}} s_i t_0 \cos\alpha_{ix} - \mu f_{jxL} = 0\right]_{jx} & jx = 1,2...,n_{jx} & (a) \\ \sum_{L=1}^{NV}\left[\sum_{i=1}^{n_{ij}} s_i t_0 \sin\alpha_{ix} - \mu f_{jyL} = 0\right]_{jy} & jy = 1,2...,n_{jy} & \text{(b)} \\ \left[\sum_{i=1}^{n_{ij}} s_i^r t_0 \cos\alpha_{ix} = 0\right]_{jx} & & (c) \\ \left[\sum_{i=1}^{n_{ij}} s_i^r t_0 \sin\alpha_{ix} = 0\right]_{jy} & & (d) \\ s_{i,\max} + s_i^r \le s_0 & & (e) \\ -s_{i,\min} - s_i^r \le s_0 & & (f) \end{cases}$$

Where t_0 is the given cross sectional area that assumed the same for all bars; s_i^r is the residual stress; s_i is the elastic stress in the i^{th} bar correspondent with the type of the L^{th} loading, $s_{i,\max} = \max(s_i)_L$, $s_{i,\min} = \min(s_i)_L$, $L = 1,2,\dots,NV$.

7 MINIMUM VOLUME OF TRUSSES AT SHAKEDOWN STATE

Similar to the problem of minimum volume of trusses at limit state the cross sectional area of bar and shakedown load are unknowns. Therefore, instead of using the condition $\min V$, we use another weak condition in the constraints: the volume of material of the truss is less than a given source of material V_0. In the formulation of the lower bound theorem, we can write:

$$\max \sum_{L=1}^{NV} \mu \left[\sum_{jx=1}^{n_{jx}} f_{jxL} + \sum_{jy=1}^{n_{jy}} f_{jyL} \right] + \min \sum_{L=1}^{NV} \left[\sum_{i=1}^{n} \frac{1}{2E} s_i^2 t_i l_i \right]_L \tag{10}$$

$$\text{s.t.:} \quad \begin{cases} \sum_{L=1}^{NV} \left[\sum_{i=1}^{n_{ij}} s_i t_i \cos\alpha_{ix} - \mu f_{jxL} = 0 \right]_{jx} & jx = 1,2...,n_{jx} \quad (a) \\ \sum_{L=1}^{NV} \left[\sum_{i=1}^{n_{ij}} s_i t_i \sin\alpha_{ix} - \mu f_{jyL} = 0 \right]_{jy} & jy = 1,2...,n_{jy} \quad (b) \\ \left[\sum_{i=1}^{n_{ij}} s_i^r t_i \cos\alpha_{ix} = 0 \right]_{jx} & (c) \\ \left[\sum_{i=1}^{n_{ij}} s_i^r t_i \sin\alpha_{ix} = 0 \right]_{jy} & (d) \\ s_{i,\max} + s_i^r \le s_0 & (e) \\ -s_{i,\min} - s_i^r \le s_0 & (f) \\ \sum_{i=1}^{n} l_i t_i \le V_0 & (g) \end{cases}$$

where t_i is the cross sectional area; s_i^r is the residual stress; s_i is the elastic stress in the i^{th} bar correspondent with type of L^{th} loading, $s_{i,\max} = \max(s_i)_L$, $s_{i,\min} = \min(s_i)_L$, $L = 1,2,\ldots,NV$.

The above formulations can be used for the problem of topology optimization of structures.

8 ALGORITHM

We use the function fmincon in the optimization toolbox of Matlab to solve the problems. The fmincon uses one of four algorithms: active-set, interior-point, SQP, or trust-region-reflective. They permit us to solve the problems of non-linear programming. In the examples, active-set algorithm is used.

9 NUMERICAL EXAMPLES

Consider a nine-bar truss, the same as one in the work of Kaliszky [4]. The geometry of the truss is described in Figure 2. The Young's modulus $E = 21000$ kN/cm^2; the yield stress $s_0 = 20$ kN/cm^2, the allowable stress, according to [4], $s_a = s_0 / 2$, is the same for all bars.

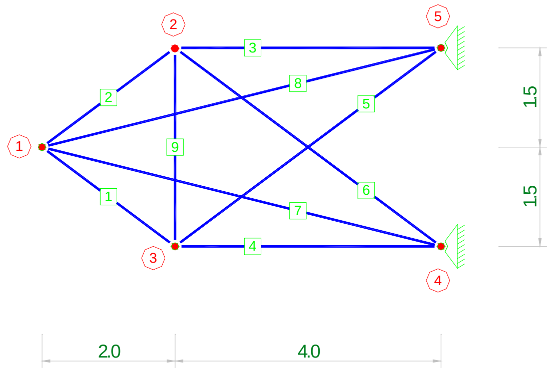

Figure 2: Nine-bar truss

9.1 Volume of the truss at elastic state

The problem is analyzed for 2 cases: the truss subjected one force and four forces

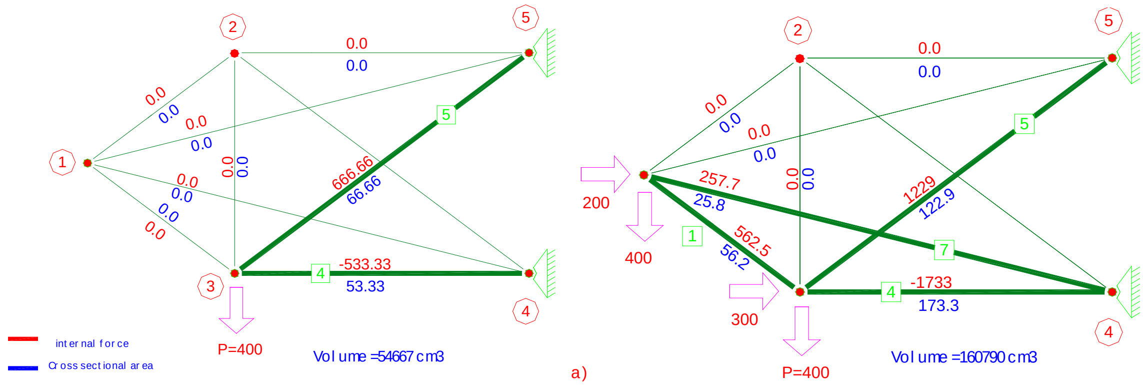

Figure 3: The results for the optimal truss at elastic state

In Figure 3, the numbers on each bar describes the internal force/cross sectional areas of the corresponding bars. In the case of single loading ($P = 400$), the bars 1,2,3,6,7,8,9 are removed. In case of multi loading the bars 2, 3, 6, 8, 9 are removed. It is noted that the distribution of the internal force ensures balance of the nodes. The solutions are *correct* because the equilibrium equations are satisfied for the nodes and the other constrains are satisfied as well.

9.2 Limit Analysis

Assuming the cross sectional area t_0 is the same for all bars in this problem. By changing t_0 (inducing a change of the volume of the truss) we have different load factors. Table 1 presents the relationship of the volume and the corresponding limit load.

Table 1. The relationship of volume and limit load

V (cm^3)	Load factor - μ	Limit Load (kN)
95923	2.1	840
78043	1.7	680
59472	1.3	520
45621	1.0	400
28182	0.7	280

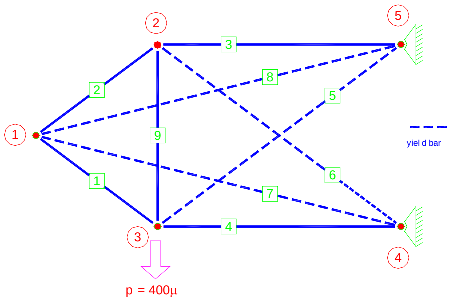

Figure 4: The limit analysis problem

The dashed bars in Figure 4 are plastic yield bars, the others are elastic ones. In this case, the load P acting on node 3, although the limit load and the corresponding volume are infinite but the number of the yield bars are the same (5,6,7,8). The equilibrium equations of the nodes and the other constraints are satisfied.

9.3 Volume of the truss at limit state

In this problem, we expect to find the limit load and the corresponding minimum of volume. By varying the source of material V_0 the limit load and the corresponding volume of truss are achieved. Specially, the volume of the truss is equal to V_0. The results shown in Table 2 indicate the relationship of volume and the corresponding limit load; the lower bound of the cross sectional area is 0 cm^2.

Table 2 The relationship of Volume and Limit load in case of single-loading (left hand side) and multi-loading (right hand side)

V_0 (cm^3)	Optimal Volume (cm^3)	Load factor μ	Limit load (kN)	V_0 (cm^3)	Optimal Volume (cm^3)	Load factor μ
57594	57594	2.1	840	166910	166910	2.1
46043	46043	1.7	680	135440	135440	1.7
34532	34532	1.3	520	103600	103600	1.3
27333	27333	1.0	400	79674	79674	1.0
15384	15384	0.6	240	47409	47409	0.6

Figure 5 describes the internal force and the corresponding area of each bar of the optimal truss. Consider the case in which the lower bound of the cross sectional area is equal to 0 cm2, the optimal truss is statically determinate and all bars yield (Figure 5a). If the lower bound of the cross sectional area is 1 cm^2, the optimal truss is indeterminate and some bars yield (Figure 5b). It is noted that the solution is correct because the equations of equilibrium are assured and the other constraints are satisfied.

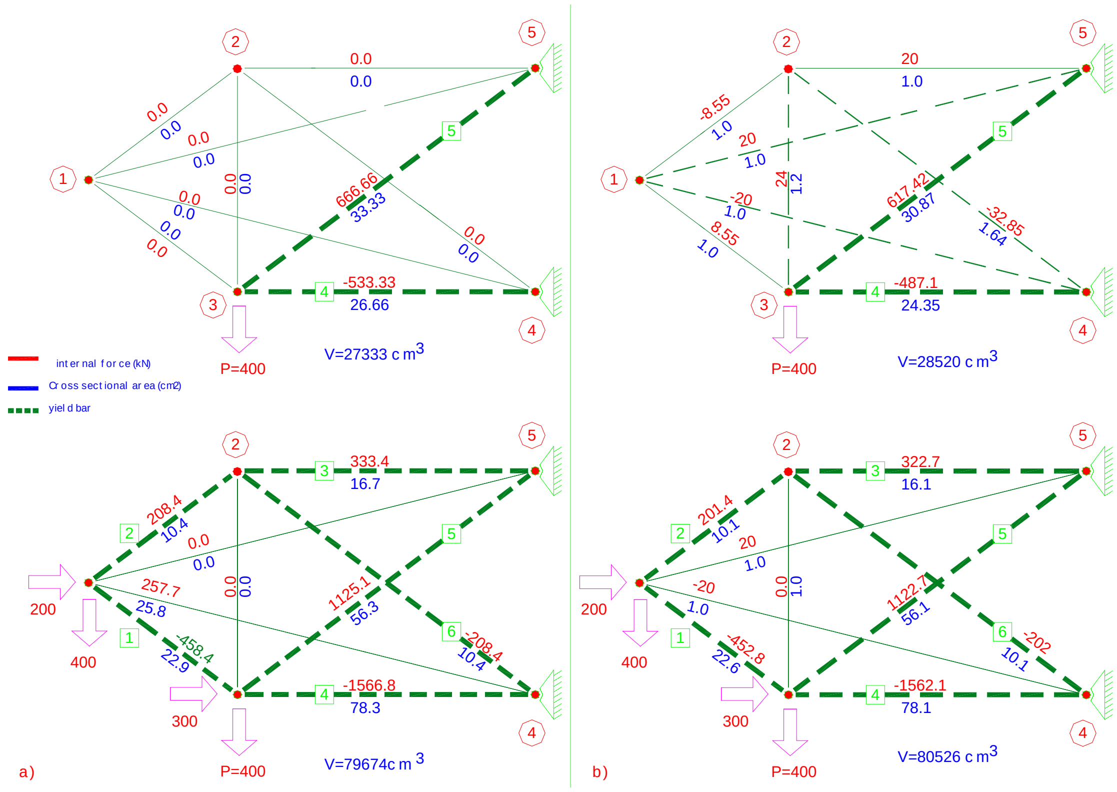

Figure 5: Distribution of internal force and cross sectional area with load factor $\mu = 1$. a) Lower bound of cross sectional area is 0 cm^2 . b) Lower bound of cross sectional area is 1 cm^2

9.4 Shakedown Analysis

Consider the truss which is subjected to two forces acting independently with the load domain as shown in Figure 6:

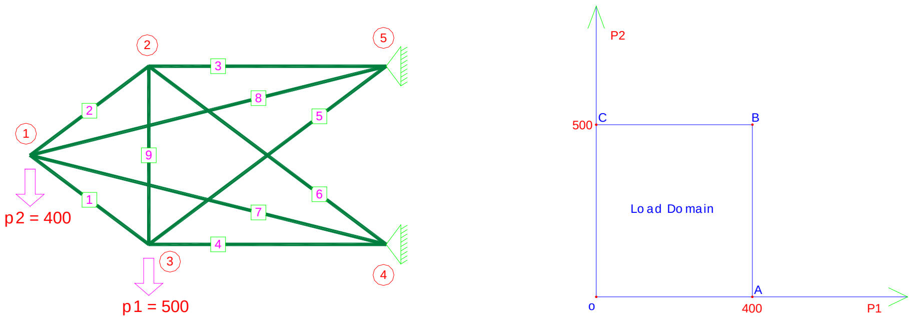

Figure 6: Example for the shakedown problem

We wish to find the shakedown load factor. The truss has nine bars with the same cross sectional area t_0. If t_0 varies then the different shakedown loads are obtained. Table 3 shows the relationship between the volume and the shakedown load. Similar to limit analysis, it can be seen that if the volume of the truss increases then the shakedown load also develops.

Table 3: Relationship between the shakedown load factor and truss volume

Load factor μ	Volume of truss (cm^3)
0	0
0.6	62542
1.0	103980
1.2	123930
1.5	154240
2.0	212180

Table 4 presents the stresses in bars of the truss with $V = 101080$ cm^3 and the load factor $\mu = 1$. The names of bars are written in the first column. The sum of elastic stress and the residual stress corresponding with A, B, C vertices of the load domain is written in the second column. The residual stresses of bars are shown in third column.

Table 4. Distribution of stress

Bar	Total stress ($\sigma_e + \sigma_r$)			R-stress (σ_r)
	C	A	B	
1	4.1063	-8.3662	-4.3975	0.132
2	-4.0959	6.1102	4.3975	-2.39
3	5.9266	11.7368	19.518	-1.8626
4	-5.9138	-11.4256	-19.518	2.1725
5	11.4986	5.9157	20	-2.5837
6	-11.5042	-8.5607	-20	-0.0617
7	-10.1499	-12.0422	-20	-2.1885
8	10.1413	13.9026	20	4.0505
9	9.36	1.4703	9.3615	1.471

The solution ensures the conditions of equilibrium and the other constraints (the total stress and the residual stress do not exceed yield stress 20 kN/cm^2).

9.5 Optimization for trusses at shakedown state

The shakedown load and the corresponding volume of truss are obtained by changing the source of material V_0. Particularly, the volume of the truss is equal to V_0. The results are shown in the Table 4 which indicates the relationship between the volume and the corresponding shakedown load and the lower bound of the cross sectional area is 0 cm^2. Similarly to the problem of optimization at limit state, we expect to find the shakedown load and the corresponding volume.

Table 4: Load factor and the corresponding volume

V_o (cm^3)	Load factor- μ	Optimal Volume (cm^3)
186090	2.0	186090
120270	1.5	120270
82302	1.0	82302
49113	0.6	49113

Table 5 presents stresses in the bars of the truss for the case in which V = 82302 cm^3 and the load factor μ=1. The names of bars are written in the first column, the sum of elastic stress and the residual stresses corresponding with A, B, C vertices of load domain are written in the second column. The residual stresses of the bars are shown in the third column. The cross sectional areas are listed in the fourth column of table. In this example, the bars 2, 3, 6, 7, 9 disappear. The other remaining bars have no residual stresses. That means that the shakedown problem becomes the limit problem.

The solutions are *correct* because: the equilibrium equations are satisfied for the nodes and the other constrains are satisfied as well.

Table 5. Distribution of stresses and the cross sectional area

Bar	Stress of bar (kN/cm^2)			Residual Stress (kN/cm^2)	Area (cm^2)
	A	B	C		
1	-0.0000	-20.0000	-20	0	25.1945
2	7.4577	13.9424	3.7669	-18.9229	0
3	-19.6636	-16.7276	-16.6604	7.7477	0
4	-9.0909	-10.9091	-20	0	73.904
5	12.5000	7.5000	20	0	67.1855
6	-18.4312	-16.0168	-19.9966	19.665	0
7	5.0853	-19.8377	19.0267	4.5554	0
8	0.0000	20.0000	20	0	20.776
9	-13.3742	15.5616	4.3138	9.8977	0

9.6 Comparison with the results of Kaliszky

Table 6 shows the comparison of the volume of the truss between our results and the ones of Kaliszky. It is worth noting that Kaliszky used some assumptions (formulas 1, 2, 3 in [4]).

Table 6. Comparison of volume of truss with Kaliszky.

		Present	Kalisky
Elastic	1 force	54667	56000
	4 forces	160790	122000
Limit	1 force	27333	29000
	4 forces	79674	78000
Shakedown	2 forces	82302	85280

10 CONCLUSION

Using stresses and the cross sectional areas as design variables gives us a simple and effective multicriteria approach to solve problems of sizing optimization of trusses. The above formulations are also employed for problems of topology optimization of trusses.

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