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LISA — a European project for FEM-based limit and shakedown analysis

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Abstract

The load-carrying capacity or the safety against plastic limit states are the central questions in the design of structures and passive components in the apparatus engineering. A precise answer is most simply given by limit and shakedown analysis. These methods can be based on static and kinematic theorems for lower and upper bound analysis. Both may be formulated as optimization problems for finite element discretizations of structures. The problems of large-scale analysis and the extension towards realistic material modelling will be solved in a European research project. Limit and shakedown analyses are briefly demonstrated with illustrative examples.

Nomenclature

$\boldsymbol{u}$	displacement
$\boldsymbol{b}$	body force
$\boldsymbol{n}$	outer normal vector
p	surface traction
P_0, T_0	reference load
t	time
$\boldsymbol{x}$	coordinate vector
$\boldsymbol{C}_i$	compatibility matrix
D, d, s, s_p, s_j	dimensions of the pipe junction
N_G	number of Gaussian points
N_V	number of load vertices
F, $\boldsymbol{f}$	yield function and matrix

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$P,\ P_{\mathrm{el}},\ P_{\mathrm{elastic}}$	actual and elastic pressure
$P(t), P_{limit}$	load and limit pressure
$P_{\mathrm{a}}, \ldots, P_{\mathrm{g}}$	actual stresses
$T,\ T_{\mathrm{el}}$	actual and elastic temperature
$\mathscr{L}$	load domain
Δ_{p}	plastic dissipation
$V,\ \partial V$	structure and its boundary
$\partial V_b,\ \partial V_N$	traction and displacement boundary
$\dot{\boldsymbol{e}}$	strain rate matrix
L	Lagrange function

Greek letters

$\alpha,\ \alpha_{\mathrm{limit}},\ \alpha_{\mathrm{Shake}}$	load, limit and shakedown factor
$\boldsymbol{\varepsilon}$	actual strain rate
$\boldsymbol{S}, \boldsymbol{S}_o$	stress and yield stress matrix
$\varepsilon^{\mathrm{P}}, \varepsilon$	plastic strain and rate
μ, μ_1, μ_2	parameters
$\boldsymbol{\rho}, \rho$	residual stress
ρ_i	discrete fictitious elastic stress vector
σ	actual stress
σ_{y}	yield stress
$\boldsymbol{\sigma}^{\mathrm{E}}$	fictitious elastic stress
$\boldsymbol{\sigma}_i^{\mathrm{E}}$	discrete fictitious elastic stress vector
$\lambda,\ \boldsymbol{\lambda}$	Lagrange factor and matrix

1. Introduction

In apparatus engineering, the design code route to plastic analysis is to a large extent an extrapolation from the elastic stress to structural failure in the sense of limit and shakedown analysis. Therefore, linear elastic calculations still form the predominant part of finite element applications. Due to fast computer development, inelastic analyses of the plastic (time-independent) or viscous (time-dependent) behaviour are increasingly used to optimize passive components for safety and for an economic operation.

Incremental analyses of the path-dependent plastic component behaviour solve the problems connected with stress assessment only partly. Besides, they are connected with relatively high computer times, personnel interaction and costs of detailed material and loading data, which cannot always be justified. The necessary data for the analysis of the detailed evolution of plastic deformations may be unavailable: the past load history cannot be determined completely afterwards and the future may not be foreseen in detail. The limit and shakedown analysis offer a method, which goes around a stress assessment and whose effort corresponds rather to the elastic calculations. Most importantly, it needs only very little key information on material behaviour and loading. Limit and shakedown analysis belong to the so-called direct or simplified methods that do not achieve the full detail of plastic structural behaviour, but instead achieve the practically essential safety margins (or load-carrying capacities) in the load space. In the design rules, the limit load concept for bending of beams has been accepted since 1947 in Great Britain and since 1959 in the USA. The concept is found in the new Eurocodes (Zeman, 1997) and applies in ductile fracture mechanics (Miller, 1988). Limit and shakedown concepts form a basis of the stress assessment concepts of all design codes for pressure vessels and pipings (Ciprian, 1980).

In view of the obvious requirement, it must be surprising that FEM-based limit and shakedown analyses have been performed so far with only few specialized university FEM programs. The industrial application failed to a large extent because of large numerical problems, which limited the method to small FEM models. In the past few years, a promising implementation in PERMAS (INTES, 1999) has been achieved at the Institute for Safety Research and Reactor Technology of Forschungszentrum Jülich GmbH. In January 1998, the 4-year Brite-EuRam project LISA was started to develop a universal limit and shakedown analysis module for the industrial FEM program PERMAS. The development will consider thermal loading, realistic material models, validation and application to complex safety and reliability problems.

2. Formulation of the problem

2.1. Criticism of stress assessment

The local capacity of the material is measured by the effective stress or yield function $f(\boldsymbol{\sigma})$; for instance, according to the hypotheses after Tresca or von Mises. Stresses $\boldsymbol{\sigma}$ are plastically admissible, if they fulfill the yield condition

$$f(\boldsymbol{\sigma}) \leq \sigma_y \qquad (1)$$

With equality in at least one point, the elastic limit is achieved and the plastification can start there. For the following considerations, the von Mises function is preferred.

Local stress gives only restricted information for the design or for the assessment of structures with respect to plastic failure. An elastic rod with a given maximum stress can still carry quite different loads up to plastic collapse depending on the loading such as bending, torsion and tension. The plastic capacity of the rod for combined loadings can be demonstrated graphically by an interaction diagram in the load space.

Obviously, fundamental differences exist between the capacity of the material measured by stresses and the plastic failure of a structure. There is no stress, which describes those limit states within the plastic regime shown in the Bree interaction diagram (Fig. 1). Therefore, in the design codes of the equipment construction, stress categories are defined so that different stresses can be assessed differently. The mechanical stresses leading to plastic collapse at limit load are called primary. Usually, the thermally induced residual stresses are called secondary. They are deformation controlled and do not influence the limit load. Thermally highly stressed components depend on this characteristic, because the possibilities for stress reduction by construction are rather limited. The increase of the wall thickness reduces the primary stresses but increase the secondary ones. Sometimes, from elastic considerations, an optimal wall thickness is recommended, with both types of stress of the same size (Smidt, 1971). This

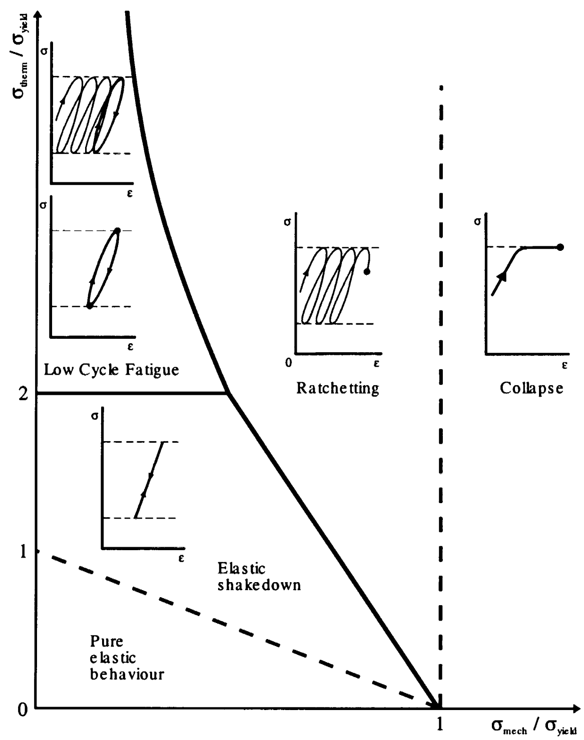

Fig. 1. Bree interaction diagram for thinwalled pipes for perfectly plastic material.

is not a rational choice in plasticity, because primary and secondary stress may have completely different consequences.

Stress calculations and stress assessments are also problematic in other regards. Practically all structures must be considered as being statically indeterminate, so that stresses $\boldsymbol{\sigma}$ can be calculated only up to some residual stress $\boldsymbol{\rho}$ from a boundary value problem. This is obvious from the equilibrium conditions of a body V under the volume loads $\boldsymbol{q}$ and under the surface loads $\boldsymbol{p}$ on the traction boundary ∂V_σ.

$$\begin{aligned} -\operatorname{div}\boldsymbol{\sigma} &= \boldsymbol{q} && \text{in } V \\ \boldsymbol{\sigma n} &= \boldsymbol{p} && \text{on } \partial V_\sigma \end{aligned} \tag{2}$$

Residual stresses ρ are in equilibrium with zero forces, i.e. they satisfy the homogeneous conditions

$$\begin{aligned} -\operatorname{div}\boldsymbol{\rho} &= \mathbf{0} && \text{in } V \\ \boldsymbol{\rho n} &= \mathbf{0} && \text{on } \partial V_\sigma \end{aligned} \tag{3}$$

Since the equations are linear, one can add them and derive also that the stresses $\boldsymbol{\sigma} + \boldsymbol{\rho}$ are in equilibrium with the same loads $\boldsymbol{q}$ and $\boldsymbol{p}$. Inherent residual stresses in welds and other residual stresses introduced during production, operation or over-loading can achieve considerable orders of magnitude. Without their knowledge, stresses remain fictitious as assessment concept.

Also, with the stability problems (buckling), the stress does not play any role in the calculation of the critical load. For plastic failure (except by instability of compression members), the critical load is calculated directly without stress assessment in the LISA approach. This could help shorten the discussions of the difficulties in classifying stress categories for complex structures (Hechmer and Hollinger, 1998).

2.2. Limit analysis

The structure V is loaded monotonously by load $\boldsymbol{P} = (\boldsymbol{q}, \boldsymbol{p})$. The engineer is interested in the load factor $\alpha > 1$, by which $\boldsymbol{P}$ can be increased up to the collapse at $\boldsymbol{\alpha P}$. As long as local flow is limited by surrounding elastic material (contained flow), no collapse occurs. The limit load theory analyzes only the collapse state, in which the structure fails with unrestricted flow without any load increase. These theorems answer the question, when the structure from ductile material is safe against collapse and when it fails with collapse.

2.2.1. Static theorem of the safe load

A structure does not collapse under a load $\alpha_s \boldsymbol{P}$ if an admissible stress field $\boldsymbol{\sigma}$ can be found, which is in equilibrium with $\alpha_s \boldsymbol{P}$. In plasticity, a stress is admissible if it satisfies the yield condition (Eq. (1)).

$$\begin{aligned} f(\boldsymbol{\sigma}) &\le \sigma_y && \text{in } V \\ -\operatorname{div}\boldsymbol{\sigma} &= \alpha_s \boldsymbol{q} && \text{in } V \\ \boldsymbol{\sigma n} &= \alpha_s \boldsymbol{p} && \text{on } \partial V_\sigma \end{aligned} \tag{4}$$

For each stress field $\boldsymbol{\sigma}$, which fulfills the conditions of the static theorem, α_s is a safety factor, so that the load-carrying capacity of the structure is not yet exhausted. One is interested in the largest factor, for which the structure does not collapse.

If one assumes the associated flow rule, then the plastic strain rates $\boldsymbol{\varepsilon}^P$ are calculated with the indefinite plastic multiplier $\lambda \ge 0$ (with $\lambda = 0$ for elastic points, i.e. for $f(\boldsymbol{\sigma}) < \sigma_y$) in accordance with

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \lambda \frac{\partial f(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} \tag{5}$$

The yield function f is positively homogeneous with the degree one, so that Euler's partial differential equation holds:

$$\frac{\partial f(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} : \boldsymbol{\sigma} = f(\boldsymbol{\sigma}) \tag{6}$$

Thus, the (plastically) dissipated specific power ε can be calculated in the collapse state. The structure collapses finally with constant stresses and therefore, with Hooke's law, the elastic strains are constant. The elastic strain rates disappear, so that $\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^P$ applies. With Eqs. (5) and (6), one obtains the dissipation density for the collapse state ($f(\boldsymbol{\sigma}) = \sigma_y$):

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}:\boldsymbol{\sigma} = \dot{\boldsymbol{\varepsilon}}:\boldsymbol{\sigma} = \lambda \frac{\partial f(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}}:\boldsymbol{\sigma} = \lambda f(\boldsymbol{\sigma}) = \lambda \sigma_{\mathrm{y}} \tag{7}$$

The collapse can be characterized by the situation that the internal (plastic) dissipation $D_{\mathrm{p}}(\dot{\boldsymbol{\varepsilon}}) > 0$ is smaller than the power $\dot{w}(\alpha_k \boldsymbol{P}) = \alpha_k \dot{W}(\boldsymbol{P}) > 0$ of the exterior loads $\alpha_k \boldsymbol{P}$:

$$D_{\mathrm{p}}(\dot{\boldsymbol{\varepsilon}}) = \int_V \dot{\boldsymbol{\varepsilon}}:\boldsymbol{\sigma}\, \mathrm{d}V = \int_V \lambda \sigma_{\mathrm{y}}\, \mathrm{d}V$$
$$< \alpha_k \left[\int_V \boldsymbol{q}\dot{\boldsymbol{u}}\, \mathrm{d}V + \int_V \boldsymbol{p}\dot{\boldsymbol{u}}\, \mathrm{d}A \right] = \alpha_k \dot{W}(\boldsymbol{P}) \tag{8}$$

This yields to the formulation of the following theorem.

2.2.2. *Kinematic theorem of the exhausted load-carrying capacity*

A structure must collapse under a load $\alpha_k \boldsymbol{P}$, if a kinematically admissible velocity field $\dot{\boldsymbol{u}}$ can be found, such that the internal dissipation $D_{\mathrm{p}}(\dot{\boldsymbol{\varepsilon}})$ is smaller than the power $\alpha_k \dot{W}(\boldsymbol{P})$ of the exterior load. A velocity field is kinematically admissible if it satisfies the compatibility conditions for the strain rates and the kinematic boundary conditions on the displacement boundary ∂V_{u}.

$$\dot{\varepsilon} = \frac{1}{2}[\nabla \dot{\boldsymbol{u}} + (\nabla \dot{\boldsymbol{u}})^T] \qquad \text{in } V \tag{9}$$

$$\dot{\boldsymbol{u}} = \dot{\boldsymbol{u}}^0 \qquad \text{on } \partial V_{\mathrm{u}}$$

$$\frac{D_{\mathrm{p}}(\dot{\boldsymbol{\varepsilon}})}{\dot{W}(\boldsymbol{P})} < \alpha_k$$

For each load P and each velocity field $\dot{u}$, which fulfill the conditions of the kinematic theorem, α_k is an overload factor, so that the load-carrying capacity of the structure is already exhausted. One is interested in the smallest factor, for which the structure collapses.

Although both theorems are directly obvious, for their proof it is needed that the yield function $f(\boldsymbol{\sigma})$ is convex after Drucker's stability postulate. The classical theory is geometrically linear and assumes associated flow. The extension to different models for hardening or damaged materials, temperature loads and temperature-dependent mechanical properties is possible.

Due to the following theorem, the limit load factor α can be calculated with the desired precision. Each safety factor and each overload factor forms a lower and an upper bound of the limit load factor, respectively.

2.2.3. *Theorem of the uniqueness of the limit load*

If α_s and α_k are a safety and an overload factor, which fulfill the conditions of the static and the kinematic theorem, respectively, then the following estimation applies to the limit load factor α.

$$\alpha_s < \alpha < \alpha_k \tag{10}$$

and

$$\sup \alpha_{\mathrm{s}} = \alpha = \inf \alpha_k \tag{11}$$

One calculates a lower bound of the limit load factor α as the largest safety factor from

$$\text{Maximize} \quad \alpha_{\mathrm{s}} \tag{12}$$

such that the conditions of (4) hold

or an upper bound of α as the least overload factor from

$$\text{Minimize} \quad \alpha_k \tag{13}$$

such that the conditions of (9) hold

Both are non-linear (infinite) optimization problems. The unknowns of the lower bound problem are the static quantities α_{s} and $\boldsymbol{\sigma}$. The kinematic quantities $\boldsymbol{\varepsilon}$ and $\dot{\boldsymbol{u}}$ are the unknowns of the upper bound problem. The complete solution of the plastic structural behaviour must satisfy both the constitutive equations as well as the static and kinematic conditions. Limit load theorems solve only one part of the complete problem of the plastic structural analysis. Their prediction is reduced also to the accurate limit load.

The stresses in a (statically undetermined) structure depend on the elastic constants and they may depend on the load history in the inelastic regime. However, the elastic material constants do not occur in the limit load theorems and, therefore, there is no causal connection between the stresses in a structure and the collapse load. It has been proven theoretically and experimentally (Maier-Leibnitz, 1928) that the stationary residual stresses do not have any influence on the limit load, if they do not modify the geometry and the yield function. Only in this regard may they remain unconsidered (as secondary stresses; this distinction between primary and secondary stress is only useful with respect to monotone loading). Also, the load history preceding the collapse is without influence. The described behaviour becomes physically understandable, if one remembers that the collapse takes place from a statically determined state.

The limit load problem is linear with respect to the allowable stress. A rigid perfectly plastic material model gives the exact limit load, because the elastic constants do not enter the problem and could be chosen arbitrary. For hardening material, the yield stress σ_y can be replaced by any higher admissible stress. Alternatively, an allowable stress may be chosen from some design codes as it is been in the sample applications. Thus, the stress assessment becomes part of the problem formulation and is globally taken into account. Local stress and local yielding can hardly be connected to global structural failure.

2.3. Shakedown analysis

Depending on the magnitude of loading, a structure can show the structural responses symbolized in Fig. 1. In addition to the plastic collapse, the structure can fail plastically with time-variant loads through:

- incremental collapse by accumulation of plastic strains over subsequent load cycles (also termed ratchetting, progressive plasticity);
- plastic fatigue by alternating plasticity in few load cycles (also termed low cycle fatigue (LCF), plastic shakedown).

The structure does not fail plastically if, finally, all plastic strain rates vanish and the dissipated energy remains finite. One says that the structure adapts to the load or it shakes down elastically. After few initially plastic cycles, no difference to the purely elastic behaviour can observed in structural mechanics quantities. The possible structural responses are symbolically illustrated in Fig. 1.

The time history of a load $\boldsymbol{P}(t) = [\boldsymbol{q}(t), \boldsymbol{p}(t)]$ is often not well known. It can, however, usually be stated that the loads vary only within a certain convex domain $\mathscr{L}$. Typically, $\mathscr{L}$ is given by amplitudes or admissible bounds. If N_L is the number of independent loads vertices $\boldsymbol{P}_1, \ldots, \boldsymbol{P}_{N_L}$ (e.g. mechanical and thermal load), then all loads $\boldsymbol{P}(t) \in \mathscr{L}$ can be represented by N_L generating loads:

$$\boldsymbol{P}(t) = \lambda_1(t)\boldsymbol{P}_1 + \ldots + \lambda_{N_L}(t)\boldsymbol{P}_{N_L}$$

$$0 \le \lambda_j(t) \le 1$$

The load-carrying capacity is exhausted by enlargement of $\mathscr{L}$ with the factor $\alpha > 1$ causing ratchetting, LCF or collapse. The shakedown theory analyzes only the shakedown state. The shakedown theorems answer the question, whether a structure from ductile material is plastically safe or not.

The same conditions as in the limit load theorems must be satisfied simultaneously at all times. Their examination in infinitely many instants is impossible and, in addition, unnecessary. One can show that it is sufficient to satisfy the shakedown conditions only in the N_L basis loads $\boldsymbol{P}_1, \ldots, \boldsymbol{P}_{N_L}$ of $\mathscr{L}$ since the shakedown theorems lead to convex optimization problems. It is not sufficient to examine the critical load cases independently, because the shakedown analysis of $\mathscr{L}$ and the limit analysis of the critical load cases (vertices of $\mathscr{L}$) give different results.

Generally, a structure under a load domain $\mathscr{L}$ shakes down if, for each load in $\mathscr{L}$, an admissible stress field, which is in equilibrium with this load, can be found. Different to limit analysis, the shakedown theorems are more complicated for hardening material. Therefore, we restrict our presentation to perfectly plastic material here.

2.3.1. Static shakedown theorem

A structure shakes down under a load domain $\alpha_s\mathscr{L}$, if for any basis load $\alpha_s\boldsymbol{P}_j$ an admissible stress field $\boldsymbol{\sigma}_j$ can be found, which is in equilibrium with $\alpha_s\boldsymbol{P}_j$.

$$\begin{array}{lll} f(\boldsymbol{\sigma}_j) \le \sigma_y & \text{in } V & j = 1, \ldots, N_L \\ -\operatorname{div}\boldsymbol{\sigma}_j = \alpha_s\boldsymbol{q}_j & \text{in } V & j = 1, \ldots, N_L \\ \boldsymbol{n}^T\boldsymbol{\sigma}_j = \alpha_s\boldsymbol{p}_j & \text{on } \partial V_\sigma & j = 1, \ldots, N_L \end{array} \tag{14}$$

2.3.2. Kinematic theorem on non-shakedown

A structure cannot shakedown under a load domain $\alpha_k\mathscr{L}$, if a kinematically admissible velocity field $\dot{u}(t)$ can be found, so that the internal (plastic) dissipation of the entire load is smaller than the work of the exterior load $\alpha_k p(t) \in \alpha_k \mathscr{L}$, i.e.

$$\dot{\boldsymbol{\varepsilon}}(t) = \frac{1}{2}[\nabla \dot{\boldsymbol{u}}(t) + \nabla \dot{\boldsymbol{u}}(t)^T] \qquad \text{in } V \tag{15}$$

$$\dot{\boldsymbol{u}}(t) = \dot{\boldsymbol{u}}^0(t) \qquad \text{on } \partial V_{\mathrm{u}}$$

$$\int_0^\infty \int_V D_{\mathrm{p}}(\dot{\boldsymbol{\varepsilon}}(t))\,\mathrm{d}V\,\mathrm{d}t$$

$$< \varepsilon_k \int_o^\infty \left[\int_V \boldsymbol{q}(t)\dot{\boldsymbol{u}}(t)\,\mathrm{d}V + \int_{\partial V_\sigma} \boldsymbol{p}(t)\dot{\boldsymbol{u}}(t)\,\mathrm{d}A \right] \mathrm{d}t$$

The shakedown factor α can be calculated exactly with $\sup \alpha_{\mathrm{s}} = \alpha = \inf \alpha_k$.

Limit and shakedown theorems have a much wider application than what could be presented in this paper. They have been generalized for the Besseling overlay hardening model (Zhang, 1991), for the two-surface plasticity model (Heitzer et al., 2000), for damaged materials (Hachemi and Weichert, 1998), for specialized engineering problems such as two-phase poroplastic solids with non-associated flow (Cocchetti and Maier, 2000) and for other material models. The development of shakedown theories for finite displacements is particularly important for shell structures, but may need further research (Weichert, 1990).

3. Objectives and intentions of LISA

3.1. Task

If one wishes to shift the design requirements into the plastic area, then one must be able to calculate safety margins to the different complex failure modes (collapse, ratchetting, LCF). Figs. 1 and 3 show examples of different structural limit states in a two-dimensional load space. Such interaction diagrams are used for steel structures and in the equipment construction (Ng and Moreton, 1982), in order to estimate the shakedown or collapse regime from an elastic calculation. They are available, however, only rarely and the elastic regime represents a strong restriction for the presented structures.

Safety against plastic failure can only partly be calculated with incremental plastic FEM analyses, by simulating the structural behaviour. It is possible to examine whether, for one load history, shakedown occurs or not. However, safety would have to be considered with respect to all possible adjacent load histories. That is possible only if one can conclude from the shakedown theory that the safe load is independent of the details of load history. Moreover, the incremental analysis is very time consuming, because the structural behaviour stabilizes only asymptotically near the safety limits. Ratchetting can be excluded often only after 40–100 load cycles. The determination of all details of the material law is hardly possible for financial and practical reasons. In limit load and shakedown analyses, the values for yield stress (plus an admissible stress above yield for hardening material) are sufficient.

For more than two decades, one has tried to make use of the large advantages of the limit and shakedown analyses with FEM discretizations of the static or kinematic theorems (Cohn and Maier, 1979). The outcome was many developments for academic research or for application to special structures (Save et al., 1991). Reflecting an obvious need for the design and assessment of pressure vessels and piping, some users just began to implement their own procedures for the static or kinematic theorems into the industrial FEM programs PERMAS (Staat and Heitzer, 1997a,b), CASTEM 2000 (Plancq et al., 1997), CODE_ASTER® (Voldoire, 2000), ABAQUS (Buckthorpe and White, 1993; Ponter and Engelhardt, 2000), ADINA (Siebler, 1998), ANSYS (Hamilton et al., 1998) and BERSAFE (Ponter and Carter, 1997). However, several practical limitations still need to be resolved in addition.

These implementations show that it is about the right time for a new development effort towards large-scale limit and shakedown analyses. Therefore, the first author initiated the Brite-EuRam project ‘LISA: FEM based limit and Shakedown analysis for Design and Integrity Assessment in European Industry’, with the objective to develop an industrially tested limit and shakedown analysis module with general applicability on complex engineering problems. The PERMAS implementation was chosen as a starting point. The new software will clearly go beyond the state-of-the-art by combining important features not found simultaneously elsewhere: non-linear kinematic hardening, damaged materials, unified upper and lower bound analysis, probabilistic analysis, and an optimized implementation for large FEM models. This will be achieved by the joint competence of a European consortium consisting of university institutes (RWTH Aachen, Liège (ULg)) and leading industrial companies (INTES, Stuttgart; Siemens, Erlangen; Electricité de France (EDF), Clamart; Bureau Veritas (BV), Paris) coordinated by Forschungszentrum Jülich (FZJ).

3.2. Limit and shakedown analysis as optimization problems

The limit load and shakedown theorems formulated for the continuum can be discretized with the FEM or they can be deduced directly for a discretized structure. The discretization of the equilibrium conditions (Eq. (2)) read with the matrix $\mathbf{C}$, the matrix $\mathbf{s}^t = (\mathbf{s}_1^T, \ldots, \mathbf{s}_i^T, \ldots, \mathbf{s}_{N_G}^T)$ of the stresses in the N_G Gauss points and the column matrix $\boldsymbol{P}$ of the exterior loads:

$$\mathbf{Cs} = \mathbf{P} \tag{16}$$

Similarly, the yield stresses in the Gauss points are combined into $\mathbf{s}_0$. The static limit load theorem leads to the optimization problem:

$$\text{Maximize} \quad \alpha_s \tag{17}$$

$$\text{such that} \quad \boldsymbol{f}(\mathbf{s}) \leq \mathbf{s}_0 \qquad \mathbf{Cs} = \alpha_s \mathbf{P}$$

The discretization of the kinematic conditions in Eq. (9) with N_F degrees of freedoms is described with the column matrices of the nodal point velocities $\dot{\boldsymbol{u}}^T = (\dot{\boldsymbol{u}}_1^T, \ldots, \dot{\boldsymbol{u}}_{N_F}^T)$ and of the strain rates $\dot{\boldsymbol{e}} = (\dot{\boldsymbol{e}}_1, \ldots, \dot{\boldsymbol{e}}_{N_G})$ by the linear set of equations

$$\mathbf{C}^T \dot{\mathbf{u}} = \dot{\mathbf{e}} \tag{18}$$

The kinematic limit load theorem leads then to the optimization problem

$$\text{Minimize} \quad \boldsymbol{\lambda}^T \boldsymbol{s}_0 \ \ (= \alpha_k) \tag{19}$$

$$\text{such that} \quad \boldsymbol{\lambda} \geq \mathbf{0} \qquad \dot{\mathbf{u}}^T \mathbf{p} = 1 \qquad \mathbf{C}^T \dot{\mathbf{u}} = \dot{\mathbf{e}}$$

These nonlinear optimization problems are dual in the sense described in Appendix A, and one formulation may be obtained from the other. Moreover, duality proves that the limit load can be calculated uniquely and with any desired accuracy (see Appendix A). The shakedown theorems lead to similar optimization problems: static theorem (Staat and Heitzer, 1997a,b), and kinematic theorem (Yan and Nguyen, 2000). For von Mises yield function, both theorems lead to non-linear optimization problems, whereas the Tresca material leads to the simpler linear optimizations problems.

In the form already given, the optimization problems cannot be represented in current FEM programs because, for example, the matrix $\mathbf{C}$ is used only on element level for the calculation of the stiffness matrix and is generally not built as a global matrix. Besides, large-scale optimization problems are numerically hard problems; i.e. in worst case analysis, the numerical effort explodes exponentially with problem size. It is thus understandable that, so far, academic applications of the limit and shakedown analysis have usually been published.

3.3. Solution of the optimization problems in LISA

The FEM discretization leads to a very large number of variables and equality and inequality constraints. Recently, some promising algorithms have been developed for the static and kinematic theorems, which may have the potential to solve numerical problems of FEM-based limit and shakedown analyses. The particular advantages of the LISA project are based on the fact that experiences with advanced algorithms are contributed by different project partners and, for the first time, the participation of an industrial FEM developer suggests an effective integration of the new developments including the just as important software quality assurance.

For the static theorems for the calculation of lower bounds, so-called basis reduction methods were developed (Zhang, 1991), extended to volume finite elements and already implemented in PERMAS (Staat and Heitzer, 1997a,b; Heitzer, 1999). So-called methods of fictitious elastic materials were developed for the kinematic theorems for the calculation of upper bounds. Upper bound methods are available to the LISA project in CODE_ASTER® (Voldoire, 2000) and in a university FEM program (Yan, 1999). From the already executed comparisons, no unique conclusions on effectiveness and robustness of the algorithms could be made.

The static methods have the advantage that they can supply safe solutions, because the numerical method achieves the optimum from below up to a termination error. Generally, these solutions are only quasi lower bounds, because all industrial FEM programs use a displacement approach and thus only the kinematic conditions can be fulfilled strictly. However, it is to be considered that, with the FEM, all conditions can be checked only in discrete points. Theoretically, one must consider the constraints in each point of the continuum, so that the solutions have a character of true bounds. That is impossible for any discretization and could become a particular problem to be considered with local failure by LCF.

The kinematic formulation leads to an objective function that is not differentiable at the boundary between the elastic and the plastic area. Therefore, the plastic dissipation (objective function of the kinematic approach) must be regularized (Yan, 1999; Voldoire, 2000), or a solution with less developed algorithms for non-smooth optimization must be used (Zhang and Bischoff, 1988).

Regardless of this discussion, the LISA project assumes that a lower and an upper bound for the limit or shakedown load factor must be provided to the user. This will be featured by the duality between static and kinematic theorems, represented in Appendix A. Then the user can estimate the numerical termination from $\alpha_k - \alpha_s$.

The non-linear structural reliability analysis likewise planned in LISA can only be performed effectively on the basis of direct limit and shakedown procedures (Heitzer, 1999; Heitzer and Staat, 2000a,b). The extensions intended for the material modeling for non-linear hardening (Zhang, 1991) and for damage (Hachemi and Weichert, 1998) is work still in progress. The LISA project could be continued with the development of a plastic structural optimization, which can generate deviating results from the elastic analysis view. This is hardly feasible today without limit and and shakedown analysis (Yang, 1993).

4. Demonstration of the already implemented static theorems

The static theorems have already been implemented in PERMAS Version 4 (Staat and Heitzer, 1997a,b; Heitzer, 1999) (PERMAS Version 7 is the current one). The application of the FEM module called PERMAS-LISA is demonstrated with two examples from the equipment construction. Further examples are shown in Staat and Heitzer (1997a,b) and Heitzer (1999). The comparison with implementations of the kinematic theorems have been performed as internal benchmarks of the LISA. In addition, examples from ductile fracture mechanics (Heitzer and Staat, 2000a,b Staat et al., 2000) and from reliability analyses (Heitzer and Staat, 2000a,b) have also been performed. For validation purposes, extensive catalogs with limit load solutions are available (Miller, 1988; Save, 1995). On the other hand, there are only few analytic or experimental results to the shakedown theory. Lang et al. (2001) reports on the shakedown experiments executed for thermal loading in LISA.

4.1. Pipe junction

The PERMAS test example of a pipe junction is examined first. The pipe junction is loaded by internal pressure and axisymmetrical quasi-steady cycles of the inner wall temperature T_i. The outer wall temperature T_a corresponds to the ambient temperature, which is set equal zero, so that the temperature difference is $T_i - T_a = T_i$. We limit ourselves to temperature-independent material data (Young's modulus, Poisson ratio and yield stress).

The junction is discretized by 125 three-dimensional 27-node elements (PERMAS element type HEXEC27). The element stresses are used in the eight vertices of the finite elements, so that 1000 nodal stresses are available (later in the LISA project, the stress will be checked at the Gauss points). The FE mesh is shown in Fig. 3. The geometrical dimensions of the junction are: internal pipe diameter, $D = 39$ mm; internal nozzle diameter, $d = 15$ mm; wall thickness of pipe and nozzle, $s = 3.44$ mm.

The following load ranges are considered.

1. The pressure P and the temperature T_i vary simultaneously with a proportionality factor (one-parameter load):

$$0 \leq P \leq \alpha\mu P_0 \qquad 0 \leq \mu \leq 1$$

$$0 \leq T_i \leq \alpha\mu T_0 \qquad 0 \leq \mu \leq 1$$

2. The pressure P and the temperature T_i vary independently (two-parameter load):

$$0 \leq P \leq \alpha\mu_1 P_0 \qquad 0 \leq \mu_1 \leq 1$$

$$0 \leq T_i \leq \alpha\mu_2 T_0 \qquad 0 \leq \mu_2 \leq 1$$

P_0 and T_0 are a reference pressure and a reference temperature in each case.

Both load domains are considered for the two cases $T_0 \geq 0$ and $T_0 \leq 0$. These correspond to the cases of an increased inside and outside temperature. The collapse pressure is obtained as special case at $T_0 = 0$. It can be compared with the collapse pressure after the AD Merkblatt B9 (Staat and Heitzer, 1997a,b). The pressure at initial yield is $P_{\text{elast}} \approx 0.0476\sigma_y$. The collapse pressure is given as $P_{\text{limit}} \approx 0.136\sigma_y = 2.85 P_{\text{elast}}$. The design pressure is obtained with the safety factor 1.5 through $P_{\text{design}} = P_{\text{limit}}/1.5 = 1.9 P_{\text{elast}} = 0.0904\sigma_y$. The limit analysis by means of PERMAS-LISA results in the limit load factor:

$$\alpha_{\text{limit}} = 2.82 \tag{20}$$

The corresponding collapse pressure is

$$P_{\text{limit}} = \alpha_{\text{limit}} P_{\text{elast}} = 0.134\sigma_y \tag{21}$$

The safety against collapse is computed about ten times faster with limit analysis than with the conventional incremental analysis. The convergence is represented in Fig. 2 as computing times in relation to the elastic analysis step. The CPU time additionally needed for the shakedown analysis with cyclic internal pressure amounts to approximately the double of the elastic calculation. In this case, the shakedown analysis converges faster than the limit analysis and compares favourable with the computing time of a elastic calculation.

The results of the shakedown analysis are represented in Fig. 3. The dashed line in Fig. 3 limits the elastic load range for one-parameter loading. The corner in the elastic limit curve in the first quadrant means that a small temperature load enables an increase of the possible elastic pressure. During two-parameter loading, the load range reaches the elastic limit for the values 1 and –1, respectively, on the abscissa and on the ordinate to the elastic limit, so that the elastic load range is cut off here. The possible elastic enlargement of the load range is then limited with the purely elastic pressure and the purely elastic temperature.

Thermal loads cause only self stresses and it follows that the thermal load does not influence the limit pressure of the junction. The limit load represents a straight line parallel to the ordinate in the interaction diagram. Abscissa and ordinate are scaled with the elastic pressure and the elastic reference stress resulting from the mechanical or thermal load at first yield.

The dash–dotted line limits the shakedown area for a one-parameter load. The shakedown factor α_{shake} related to the elastic solution is close to 2 in the entire area ($1.98 \leq \alpha_{\text{shake}} \leq 2$). For proportional loading, the elastic shakedown range is obtained by a linear expansion of the elastic operation range by the factor 2. This is caused by the local failure of the junction in the connecting piece edges by LCF. The point of first yield and the local failure point coincide. In this case of local failure, the shakedown loads do not increase for linear and non-linear kinematic hardening material, as shown theoretically in Zhang (1991), Heitzer (1999). The full line limits the shakedown area for a two-parameter load. The shakedown factor with respect to the elastic solution varies between 1.46 and 2. The example shows that the

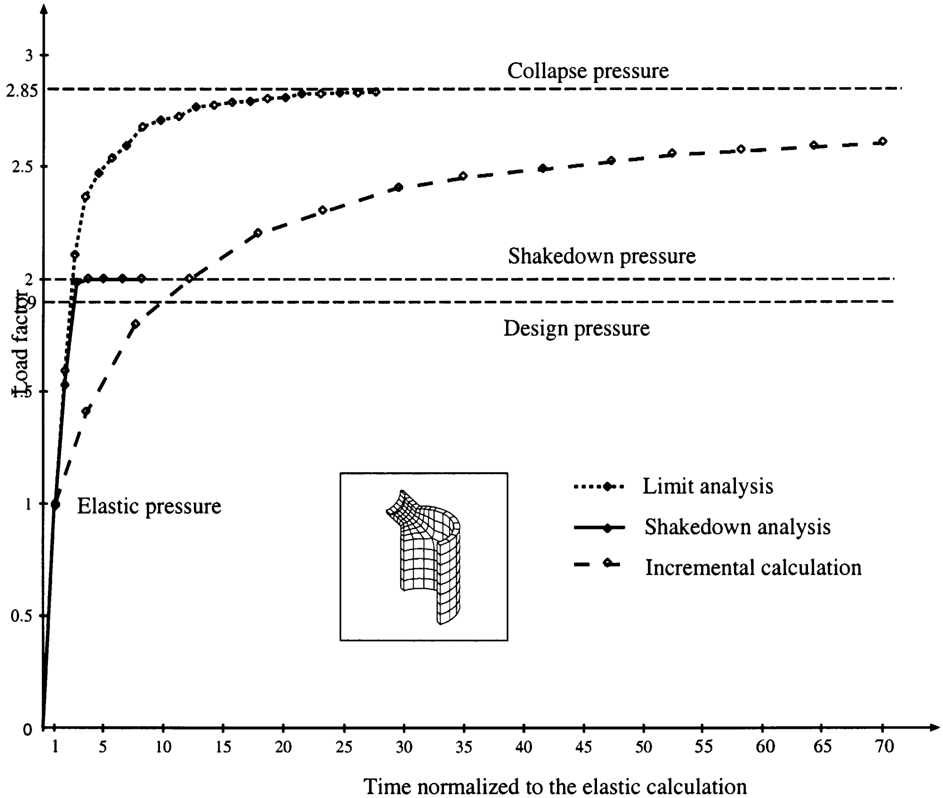

Fig. 2. CPU time for the pipe junction under internal pressure.

elastic area limits the operation of the component strongly already under pure internal pressure.

4.2. Torispherical pressure vessel head

In Japan, 16 teams executed a benchmark program for the incremental limit load calculation with different FEM programs and discretizations (see Yamamoto et al., 1997; Table 2). The examined model is a pressure vessel consisting of a cylinder and a torispherical head with a conical transition under internal pressure. The problem was chosen because the ASME code does not strictly apply to such a vessel design. Elastic perfectly plastic material behaviour was assumed in all limit load analyses. In the example, the structural steel SFVQ1A of the Japanese Industry Standard (JIS) is selected (corresponding to the heat-resistant steel 20MnMoNi45 according to DIN). The material data are calculated after the MITI code for a temperature of 300°C. The dimensions and material data for the benchmark example are summarized in Table 1. The reduced (by a factor 2) material parameter $S_m \approx 0.5\sigma_y$ is used instead of the yield stress σ_y.

The elastic analyses of the Japanese teams result in a pressure at yield initiation (yield pressure) of 8.6 N mm^{-2} for 1.5 mm$^{-2}S_m = 27$. The Japanese teams performed the limit analyses with the help of the Double Elastic Slope Method for $1.5S_m$ for perfectly plastic material behaviour (see ASME code, Section III, NB-3213,25). FE programs and the most important characteristics of the used FE nets are listed in Table 2. Seven FE programs were tested with different FE nets with up to 2435 nodes.

The vessel has been discretized by 208 rotationally symmetric nine-node elements (PERMAS element type QUAX9) for the limit analysis with PERMAS-LISA. The elastic analysis results in a

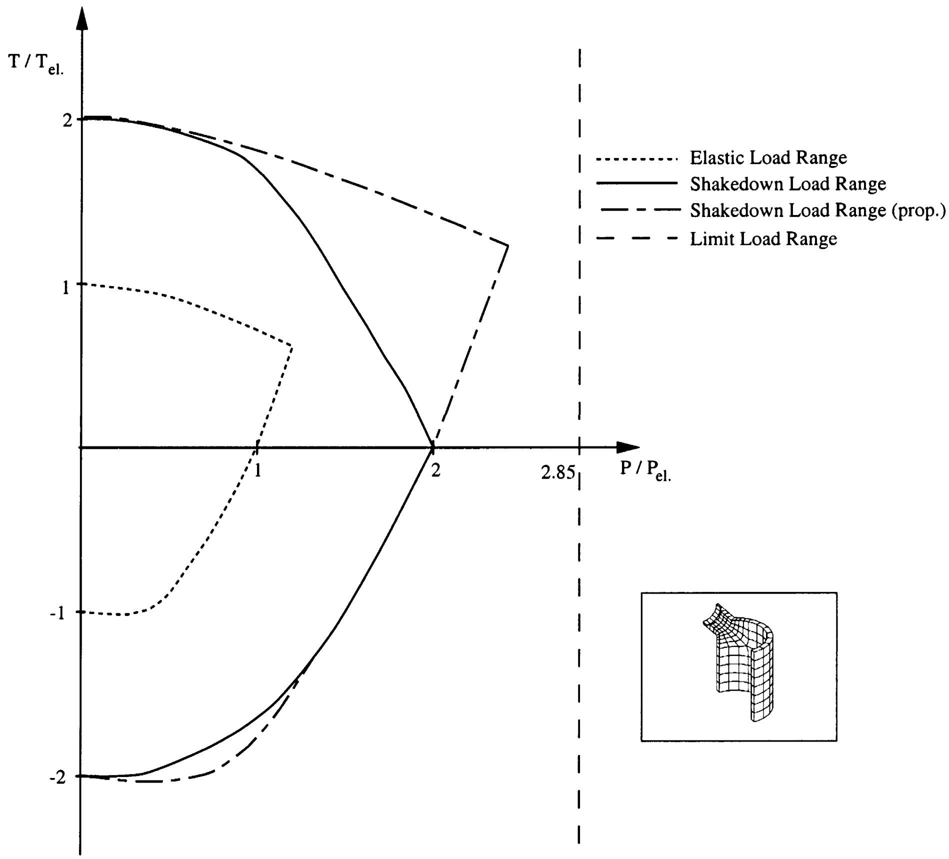

Fig. 3. Interaction diagram of the pipe junction.

yield pressure of 8.5 N mm^{-2} for the discretization represented in Fig. 4 and corresponds well with the results of the Japanese teams (see also the elastic stress shadings in Fig. 4). The theoretical yield pressure and the limit pressure for the closed cylinder $1.5S_m$ are given by (Szabo, 1972):

$$P^z_{\text{elast}} = \frac{1.5S_m}{\sqrt{3}}\left(1 - \frac{R_z^2}{(R_z+s)^2}\right) = 21.46 \text{ N mm}^{-2} \tag{22}$$

$$P^z_{\text{limit}} = 1.5S_m \frac{2}{\sqrt{3}} \ln \frac{R_z+s}{R_z} = 23.05 \text{ N mm}^{-2} \tag{23}$$

Therefore, the limit load factor α_{limit} for the cylinder is $\alpha_{\text{limit}} = 1.074$. The limit load calculated with PERMAS-LISA is 23.00 N mm^{-2}. This means a deviation of 0.2% from the limit load of the cylinder, so that the crown, the transition and the vessel head do not have considerable influence on the limit load. A comparison of the loads of the vessel and the limit load after limit analysis, obtained by the Japanese teams, is represented in

Table 1
Dimensions and material data of the vessel (Yamamoto et al., 1997)

Length of conical transitition	$l = 658.2$ mm
Length of shell	$L = 3000$ mm
Interior radius of crown	$R_b = 4500$ mm
Interior radius of knuckle	$R_k = 360$ mm
Interior radius of shell	$R_z = 3000$ mm
Vessel wall-thickness	$s = 225$ mm
Young's modulus	$E = 1.75 \times 10^5$ N mm^{-2}
Poisson's ratio	$\nu = 0.3$
Yield stress	$\sigma_y = 370$ N mm^{-2}

Table 2
List of applied codes and limit pressures (Yamamoto et al., 1997), except PERMAS-LISA

Code	Element type	Number of nodes	Number of elements	Number of elements/thickness	Limit pressure (N mm^{-2})
MARC	Eight-node Quad.	569	160	4	20.8
MARC	Eight-node Quad.	725	210	5	21.0
ABAQUS	Four-node Quad.	628	405	5	21.4
FINAS	Eight-node Quad.	579	156	3	21.0
FINAS	Eight-node Quad.	579	156	3	22.0
ADINA	Eight-node Quad.	1388	405	5	20.6
STAX	Four-node Quad.	343	288	6	21.9
PC-FEAP	Four-node Quad.	315	248	4	21.0
MARC	Eight-node Quad.	681	192	4	21.0
ABAQUS	Eight-node Quad.	849	240	4	22.0
ABAQUS	Four-node Quad.	492	405	5	21.0
FINAS	Eight-node Quad.	350	96	4	21.0
ABAQUS	Eight-node Quad.	2435	744	8	21.5
ANSYS	Four-node Quad.	492	405	5	21.8
FINAS	Four-node Quad.	310	244	4	22.9
PERMAS-LISA	Nine-node Quad.	832	208	4	23.0

Table 2. With the vessel examined in Yan (1999), the collapse takes place with the limit load of the spherical vessel head.

5. Summary and conclusions

Limit and shakedown analyses are simplified but exact methods of plasticity, which do not contain any restrictive prerequisites apart from sufficient ductility. The simplifications concern the details of material behaviour and of the load history. This implies that less data is needed for such analyses — an important advantage if such data is expensive, uncertain or unavailable in principle. Differently to the classical handling of non-linear problems in structural mechanics, the methods lead on optimization problems. The large size of the FEM models for realistic problems has delayed the industrial application of the limit and shakedown analyses.

In the Brite-EuRam project LISA, a procedure for the direct calculation of the load-carrying capacity of ductile structures is developed on the basis of the industrial FEM program PERMAS. The operation range of passive components and of buildings can be extended to the plastic regime, without increasing the efforts in relation to elastic analyses substantially. The computing time permits parameter studies and the calculation of interaction diagrams, which give a fast overview on the possible operation ranges. It is shown that, dependent on the component and its loads, important safety gains can be obtained for the extension of the operation ranges. The non-linear reliability analyses likewise developed in LISA are only possible on the basis of direct

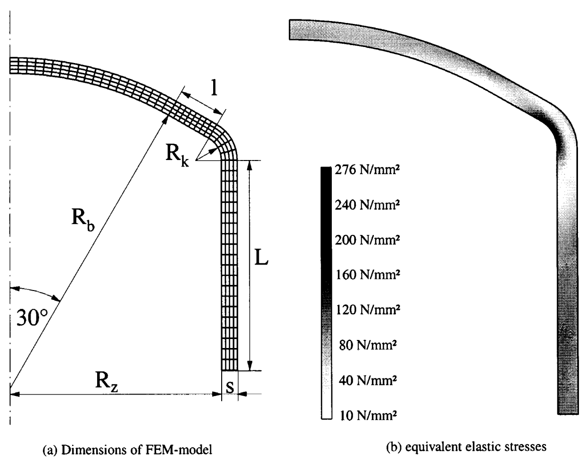

Fig. 4. Dimensions and von Mises stresses for the FEM model.

methods. A clear lack of experiments for the proof of the limits between elastic shakedown and the failure by LCF or by ratchetting must be stated.

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Appendix A. Lagrange Duality

The minimum and maximum problems derived from the static and kinematic theorems for the discretized structures are dual. In the case of limit analysis, we give proof of this statement. More detail on Lagrange Duality may be found in Bazaraa et al. (1993).

Let the lower bound problem be the primal one:

$$\text{Maximize} \qquad \alpha_s \tag{A1}$$

$$\text{such that} \qquad \mathbf{f}(s) - \mathbf{s}_0 \leq \mathbf{0} \qquad \mathbf{Cs} - \alpha_s \mathbf{P} = \mathbf{0}$$

The inequality constraints of the N_G Gauss points were collected to the vectors $\mathbf{f}$, $\mathbf{s}$ and $\mathbf{s}_0$. The unknowns are the limit load factor α_s and the stresses s. The minimum problem with restrictions is transformed into an unrestricted problem by the Lagrangian $L(\alpha_s, \mathbf{s}, \dot{\mathbf{u}}, \lambda)$, such that the optimality conditions for unrestricted problems hold. With the Lagrange factors $\boldsymbol{\lambda} \geq 0$ and $\dot{\boldsymbol{u}}$, it holds that

$$L(\alpha_s, \mathbf{s}, \dot{\mathbf{u}}, \lambda) = \alpha_s + \dot{\mathbf{u}}^T(\mathbf{Cs} - \alpha_s \mathbf{P}) - \boldsymbol{\lambda}^T(\mathbf{f}(s) - \mathbf{s}_0) \tag{A2}$$

In the minimum, the Lagrangian $L(\alpha_s, \mathbf{s}, \dot{\mathbf{u}}, \lambda)$ has a saddle point, so that the optimal value is the solution of

$$\min_{\dot{\mathbf{u}}, \lambda} \max_{\alpha_s, \mathbf{s}} L(\alpha_s, \mathbf{s}, \dot{\mathbf{u}}, \lambda) \tag{A3}$$

The necessary optimality conditions of the maximum are

$$\frac{\partial L}{\partial \alpha_s} = 1 - \dot{\boldsymbol{u}}^T \mathbf{P} = 0 \tag{A4}$$

$$\frac{\partial L}{\partial \mathbf{s}} = \dot{\mathbf{u}}^T \mathbf{C} - \lambda^T \frac{\partial \mathbf{f}}{\partial \boldsymbol{s}} = 0 \tag{A5}$$

Eq. (A4) means a normalization of the external power of loading $\dot{W}_{ex} = \dot{\mathbf{u}}^T \mathbf{P} = 1$ of the discretized structure. By substituting Eq. (A4) in the dual objective function $\max_{\alpha_s, \mathbf{s}} L(\alpha_s, \mathbf{s}, \dot{\mathbf{u}}, \lambda)$, with the Euler PDE for the homogeneous function $\boldsymbol{f}(\boldsymbol{s})$:

$$\mathbf{s}^T \frac{\partial \mathbf{f}(\boldsymbol{s})}{\partial \mathbf{s}} = \mathbf{f}(\boldsymbol{s}) \tag{A6}$$

and with $\boldsymbol{\lambda} \geq \mathbf{0}$, it follows that

$$l(\boldsymbol{\lambda}) = \max_{\alpha_s, \mathbf{s}} L(\alpha_s, \mathbf{s}, \dot{\mathbf{u}}, \lambda) = \lambda^T \mathbf{s}_0 = \dot{W}_{in}(\dot{\boldsymbol{e}}) \tag{A7}$$

Eq. (A3) is derived by Eqs. (A4), (A5) and (A7) such that the dual problem is defined by the non-smooth mathematical program:

$$\text{Minimize} \qquad \lambda^T \boldsymbol{s}_0 \ (= \alpha_k) \tag{A8}$$

$$\text{such that} \qquad \lambda \geq \mathbf{0} \qquad \dot{\boldsymbol{u}}^T \mathbf{P} = 1$$

$$\mathbf{C}^T \dot{\mathbf{u}} - \lambda^T \frac{\partial \mathbf{f}}{\partial \mathbf{s}} = 0$$

Because of the normalization $\dot{\mathbf{W}}_{ex} = \dot{\mathbf{u}}^T \mathbf{P} = 1$, it holds that $\alpha_k = l(\boldsymbol{\lambda}) = \dot{W}_{in}(\dot{\boldsymbol{\varepsilon}})$.

The Lagrange factors of the primal problem are the unknowns of the dual problem and vice versa. The dual problem is formulated in the kinematic terms $\dot{\boldsymbol{u}}$ and $\boldsymbol{\lambda}$. With

$$\dot{\boldsymbol{e}}^p = \lambda^T \frac{\partial \mathbf{f}(\boldsymbol{s})}{\partial \boldsymbol{s}} \tag{A9}$$

Eq. (A5) could be reformulated for the associated flow rule and $\dot{\boldsymbol{e}}^p = \dot{\boldsymbol{e}}$ in the collapse state

$$C^T \dot{\mathbf{u}} - \dot{\mathbf{e}} = \mathbf{0} \tag{A10}$$

which is automatically satisfied in a displacement FEM discretization. Equation Eq. (19) shows that $\boldsymbol{\lambda}$ may be replaced by the collection of effective strain rates $\dot{\boldsymbol{\varepsilon}}_{\mathrm{eq}}$, and always $\boldsymbol{\lambda} = \dot{\boldsymbol{\varepsilon}}_{\mathrm{eq}} \geq 0$. Then the dual problem reduces to

$$\text{Minimize} \quad \dot{\boldsymbol{\varepsilon}}_{\mathrm{eq}}^{T} \boldsymbol{s}_0 \ \ (= \alpha_k) \tag{A11}$$

$$\text{such that} \quad \dot{\mathbf{u}}^T \mathbf{P} = 1$$

The saddle-point properties of the Lagrangian show that the maximum problem is concave and the minimum problem is convex, such that both problems have the same optimal value:

$$\max \alpha_{\mathrm{s}} = \alpha = \min \alpha_k \tag{A12}$$

Because of the convexity of the problem, the obtained local optimum is a global one, such that the limit load factor is unique.

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